Florian Cajori

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AN

# ELEMENTARY TREATISE

ON

# ANALYTICAL GEOMETRY:

TRANSLATED FROM THE FRENCH OF J. B. BIOT, FOR

THE USE OF THE

CADETS OF THE VIRGINIA MILITARY INSTITUTE,

AT LEXINGTON, VA.;

AND

ADAPTED TO THE PRESENT STATE OF MATHEMATICAL INSTRUCTION IN THE

COLLEGES OF THE UNITED STATES.

BY

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TO THE

BOARD OF VISITERS

OF THE

VIRGINIA MILITARY INSTITUTE,

THROUGH WHOSE ENCOURAGEMENT AND SUPPORT

THIS WORK

HAS BEEN UNDERTAKEN;

AND BY WHOSE ZEAL AND WISDOM IN ORGANIZING AND DIRECTING

The Institute,

THE CAUSE OF SCIENCE HAS BEEN PROMOTED,

AND

THE INTERESTS OF THE STATE OF VIRGINIA ADVANCED.

# PREFACE.

THE original work of M. BIOT was for many years the Text Book in the U. S. Military Academy at West Point. It is justly regarded as the best elementary treatise on Analytical Geometry that has yet appeared. The general system of Biot has been strictly followed. A short chapter on the principal Transcendental Curves has been added, in which the generation of these Curves and the method of finding their equations are given. A Table of Trigonometrical Formulæ is also appended, to aid the student in the course of his study.

The design of the following pages has been to prepare a Text Book, which may be readily embraced in the usual Collegiate Course, without interfering with the time devoted to other subjects, while at the same time they contain a comprehensive treatise on the subject of which they treat.

*Virginia Military Institute,*
JULY, 1840.

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# ANALYTICAL GEOMETRY.

## CHAPTER I.

### PRELIMINARY OBSERVATIONS.

1. ALGEBRA is that branch of Mathematics in which quantities are represented by letters, and the operations to be performed upon them indicated by signs. It serves to express generally the relations which must exist between the known and unknown parts of a problem, in order that the conditions required by this problem may be fulfilled. These parts may be numbers, as in Arithmetic, or lines, surfaces, or solids, as in Geometry.

2. Before we can apply Algebra to the resolution of Geometrical problems, we must conceive of a magnitude of known value, which may serve as a term of comparison with other magnitudes of the same kind. A magnitude which is thus used, to compare magnitudes with each other, is called a *unit of measure,* and must always be of the same dimension with the magnitudes compared.

3. In Linear Geometry the unit of measure is a line, as a foot, a yard, &c., and the length of any other line is expressed by the number of these units, whether feet or yards, which it contains.

Let CD and EF be two lines, which we wish to compare with each other; AB the unit of measure. The line CD containing AB *six* times, and the line EF containing the same unit *three* times, CD and EF are evidently to each as the numbers 6 and 3.

4. In the same manner we may compare surfaces with surfaces, and solids with solids, the unit of measure for surfaces being a known square, and for solids a known cube.

5. We may now readily conceive lines to be added to, subtracted from, or multiplied by, each other, since these operations have only to be performed upon the numbers which represent them. If, for example, we have two lines, whose lengths are expressed numerically by $a$ and $b$, and it were required to find a line whose length shall be equal to their sum, representing the required line by $x$, we have from the condition,

$$x = a + b,$$

which enables us to calculate arithmetically the numerical value of $x$, when $a$ and $b$ are given. We may thus deduce the line itself, when we know its ratio $x$ to the unit of measure.

6. But we may also resolve the proposed question geometrically, and *construct* a line which shall be equal to the sum of the two given lines. For, let $l$ represent the absolute length of the line which has been chosen as the unit of measure, and A, B, and $x$ the absolute lengths of the given and required lines. The numerical values $a$, $b$, $x$, will express the ratios of these three lines to the unit of measure, that is, we have,

$$a = \frac{A}{l}, \qquad b = \frac{B}{l}, \qquad x = \frac{x}{l}.$$

These expressions being substituted in the place of $a$, $b$, $x$ in the equation

$$x = a + b,$$

the common denominator $l$ disappears, and we have

$$X = A + B.$$

Hence, to obtain the required line, draw the indefinite line AB, and lay off from A in the direction AB the distance AC equal to A, and from C the distance CB equal to B, AB will be the line sought.


7. The *construction* of an analytical expression, consists in finding a geometrical figure, whose parts shall bear the same relation to each other, respectively, as in the proposed equation.

8. The subtraction of lines is performed as readily as their addition. Let $a$ be the numerical value of the greater of the two lines, $b$ that of the less, and $x$ the required difference, we have,

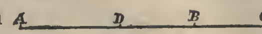
$$x = a - b,$$

an expression which enables us to calculate the numerical value of $x$, when $a$ and $b$ are known. To construct this value, substitute as before, for the numerical values $a$, $b$, $x$, the ratios $\frac{A}{l}$, $\frac{B}{l}$, $\frac{x}{l}$, of the corresponding lines to the unit of measure; the common denominator $l$ disappears, and the equation becomes

$$X = A - B,$$

which expresses the relation between the absolute lengths of

these three lines. Drawing the indefinite line AC, and laying off from A a distance AB equal to A, and from B in the direction BA, a distance BD equal to B, AD will express the difference between A and B.




9. Comparing this solution with that of the preceding question, we see by the nature of the operations themselves, that the direction of the line BD or B is changed, when the sign which affects the numerical value of B is changed. This analogy between the inversion in position of lines, and the changes of sign in the letters which express their numerical values, is often met with in the application of Algebra to Geometry, and we shall have frequent occasion to verify it, in the course of this treatise.

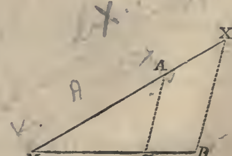
10. From the combination of quantities by addition and subtraction, let us pass to their multiplication and division. Let us suppose, for example, that an unknown line X depends upon three given lines A, B, C, so that there exists between their numerical values the following relation,

$$x = \frac{ab}{c}$$

This relation enables us to calculate the value of $x$, when $a$, $b$, and $c$ are known. To make the corresponding geometrical construction, substitute for $a$, $b$, $c$, and $x$, the ratios $\frac{A}{l}, \frac{B}{l}, \frac{C}{l}, \frac{X}{l}$, of the corresponding lines to the unit of measure; $l$ disappears from the fraction, and we have

$$X = \frac{AB}{C}$$

from which we see that the required line is a fourth proportional to the three lines A, B, C. Draw the in-



definite lines MB and MX, making any angle with each other; Lay off MC = C, MB = B, and MA = A, join C and A, and draw BX parallel to CA, MX is the required line. For, the triangles MAC, MXB, being similar, we have

$$\mathrm{MC} : \mathrm{MB} :: \mathrm{MA} : \mathrm{MX}$$

or

$$\mathrm{C} : \mathrm{B} :: \mathrm{A} : \mathrm{X}$$

and consequently $\mathrm{X} = \dfrac{\mathrm{AB}}{\mathrm{C}}$

which fulfils the required conditions.

11. In the example which we have just discussed, as well as in the two preceding, when we have passed from the numerical values of the lines, to the relations between their absolute lengths, we have seen that the unit of measure $l$ has disappeared; so that the equation between the absolute lengths was exactly the same as that between the numerical values. We could have dispensed with this transformation in these cases, and proceeded at once to the geometrical construction, from the equation in $a$, $b$, and $x$, by considering these letters as representing the lines themselves. But this could not be done in general. For, this identity results from the circumstance that the proposed equations contain only the ratios of the lines to each other, independently of their absolute ratio to the unit of measure. This will be evident, if we observe that the equations

$$x = a + b, \quad x = a - b, \quad x = \frac{ab}{c}$$

may be put under the following forms,

$$1 = \frac{a}{x} + \frac{b}{x}, \quad 1 = \frac{a}{x} - \frac{b}{x}, \quad 1 = \frac{ab}{cx}$$

which express the ratios of $a$, $b$, $c$, and $x$, with each other,

and whose form will not be changed, if we substitute for these letters the equivalent expressions $\frac{A}{l}$, $\frac{B}{l}$, $\frac{B}{l}$, $\frac{X}{l}$.

12. But it will be otherwise, should the proposed equation, besides containing the ratios of the lines A, B, C and X, with each other, express the absolute ratio of any of them to the unit of measure. For example, if we had the equation

$$x = ab$$

the numerical value of $x$ can be easily calculated, since it is the product of two abstract numbers; and this value being known, we can easily construct the line which corresponds to it. But, if we wished to pass from this equation to the analytical relation between the absolute lengths of the lines A, B, X, by substituting for $a$, $b$, $x$, the expressions $\frac{A}{l}$, $\frac{B}{l}$, $\frac{X}{l}$, $l$ being of the square power in the denominator of the second member, and of the first power in the first member, it would no longer disappear, and we should have, after reducing,

$$X = \frac{AB}{l},$$

in which the line X is a fourth proportional to the lines $l$, A, B. In this, and all other analogous cases, we cannot suppose the same relation to exist between the absolute lengths of the lines as between their numerical values; and this impossibility is shown from the equation itself. For, if $a$, $b$, and $x$, represented lines, and not abstract numbers, the product $a\ b$ would represent a surface, which could not be equal to a line $x$.

13. By the same principle, we may construct every equation of the form

$$x = \frac{a\,b\,c\,d\,\ldots\ldots}{b'\,c'\,d'\,\ldots}$$

in which $a$, $b$, $c$, $d$, $b'$, $c'$, $d'$, &c., are the numerical values of so many given lines. If we suppose the equation homogeneous, which will be the case if the numerator contain one factor more than the denominator, then substituting for the numerical values their geometrical ratios, we have

$$\mathrm{X} = \frac{\mathrm{A\ B\ C\ D}\ \ldots}{\mathrm{B'\,C'\,D'}\ \ldots}$$

But the first part $\frac{\mathrm{AB}}{\mathrm{B'}}$ may be considered as representing a line A$''$, the fourth proportional to B$'$, A, and B. Combining this line with the following ratio $\frac{\mathrm{C}}{\mathrm{C'}}$, the product $\frac{\mathrm{A''\,C}}{\mathrm{C'}}$ will represent a new line A$'''$, the fourth proportional to C$'$, A$''$, and C. This being combined with $\frac{\mathrm{D}}{\mathrm{D'}}$ would give a product $\frac{\mathrm{A'''\,D}}{\mathrm{D'}}$, which may be constructed in the same manner. The last result will be a line, which will be the value of $x$.

14. We have supposed the numerator to contain one more factor than the denominator. If this had not been the case, $l$ would have remained in the equation to make it homogeneous. For example, take the equation

$$x = a\,b\,c\,d$$

the transformed equation becomes

$$\mathrm{X} = \frac{\mathrm{ABCD}}{l^3}.$$

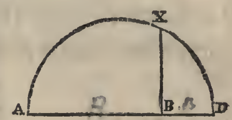
an expression which may be constructed in the same manner as the preceding.

15. Besides the cases which we have just considered, the

unknown quantity is often given in terms of radical expressions, as

$$x = \sqrt{ab}, \quad x = \sqrt{a^2 + b^2}, \quad x = \sqrt{a^2 - b^2}.$$

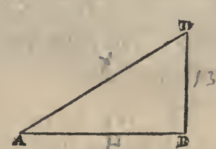
The first $\sqrt{ab}$, expresses a mean proportional between $a$ and $b$, or between the lines which these values represent. Laying off on the line AD, AB = A, BD = B, and on AD as a diameter describing the semi-circle AXB, BX perpendicular to AB at the point B, will be the value of X. For, from the properties of the circle, the line BX is a mean proportional between the segments of the diameter.




16. If we take the example,

$$x = \sqrt{a^2 + b^2}$$

it is evident that the required line is the hypothenuse of a right angled triangle, of which the sides are AB = A, and BD = B; for we have




$$\overline{AD}^2 = \overline{AB}^2 + \overline{BD}^2$$

or

$$X^2 = A^2 + B^2$$

$$X = \sqrt{A^2 + B^2}$$

17. We may also construct by the right angled triangle, the expression

$$x = \sqrt{a^2 - b^2}$$

the required line being no longer the hypothenuse, but one of the sides. Making BD = A, and DA = B, we have

$$\overline{AB}^2 = \overline{AD}^2 - \overline{BD}^2$$

or
$$X^2 = A^2 - B^2$$
$$X = \sqrt{A^2 - B^2}$$

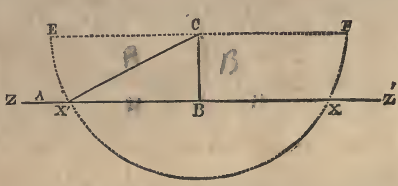
18. Let us now apply these principles to the example,

$$x^2 - 2\,a\,x = -\,b^2.$$

Solving the equation with respect to $x$, we get the two roots,

$$x = a + \sqrt{a^2 - b^2}, \qquad x = a - \sqrt{a^2 - b^2}.$$

The radical part of these expressions may be evidently represented by a side of a right-angled triangle, of which the line A is the hypothenuse, and the line B the other side. Draw the indefinite line ZZ′; at any point B erect a perpendicular, and make BC = B. From the point AC as a centre with a radius equal to A, describe a circumference of a circle, which will cut ZZ′, generally, in two points X, X′, equally distant from B. The segment BX, or BX′, will represent the radical $\sqrt{A^2 - B^2}$, and if from the point B we lay off on ZZ′, a length BA = A, $AX = \sqrt{A^2 - B^2} + A$ will represent the first value of X, and $AX' = A - \sqrt{A^2 - B^2}$ will represent the second value.




19. If B = A, it is evident that the circle will not cut the line ZZ′, but be tangent to it at B. The two lines BX and BX′ will reduce to a point, and AX and AX′ will be equal to each other, and to the line A. This result corresponds strictly with the change which the Algebraic expression undergoes; for $a = b$ causes the radical $\sqrt{a^2 - b^2}$ to disap-

pear, and reduces the second member to the first term, and the two roots become equal to $a$.

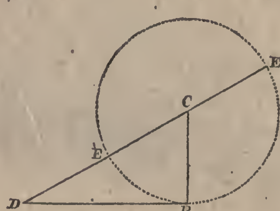
20. If B $>$ A, the circle described from the point C as a centre will not meet the line ZZ′, and the solution of the question is impossible. This is also verified by the equation, for $b > a$ makes the radical $\sqrt{a^2 - b^2}$ imaginary, and consequently the two roots are impossible.

21. If the second member of the equation had been positive, the construction would have been a little different. In this case we would have,

$$x^2 - 2\,a\,b = b^2\,;$$

and the roots would be,

$$x = a + \sqrt{a^2 + b^2}, \quad x = a - \sqrt{a^2 + b^2}.$$




Here the radical part is represented by the hypothenuse of a right-angled triangle, whose sides are A and B. Take DB = B; at the point B, erect a perpendicular BC = A; DC will be the radical part common to the two roots. If, then, from the point C as a centre, with a radius CB = A, we describe a circumference of a circle, cutting DC in E′, and its prolongation in E, the line DE will be equal to $A + \sqrt{A^2 + B^2}$, which will represent the first value of $x$, but the second segment $DE' = \sqrt{A^2 + B^2} - A$ will only represent the second root, by changing its sign, that is, the root will be represented by $-$ DE′.

22. Here the change of sign is not susceptible of any direct interpretation, since, admitting that it implies an inversion of position, we do not see how this happens, as there is

no quantity from which DE′ is to be taken. But the difficulty disappears, if we consider the actual value of $x$ as a particular case of a more general problem, in which the roots are,

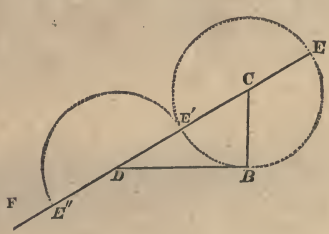
$$x = a + c + \sqrt{a^2 + b^2}, \quad x = a + c - \sqrt{a^2 + b^2}.$$

$c$, representing the numerical value of a new line, which is also given. This form of the roots would make $x$ depend upon another equation of the second degree, which would be,

$$x^2 - 2\,(a + c)\,x = b^2 - 2\,a\,c - c^2\,;$$

in which, if we make $c = o$, we obtain the original values of $x$.

In the new example, the construction of the radical part is precisely the same, for, taking DB = B and BC = A, the hypothenuse DC will represent $\sqrt{A^2 + B^2}$. From the point C as a centre with a radius equal to A, describe a circumference of a circle, $DE = A + \sqrt{A^2 + B^2}$ and $-DE' = A - \sqrt{A^2 + B^2}$.




To obtain the first root, we have only to add C to DE, which is done by laying off DF = C, and FE will represent $C + A + \sqrt{A^2 + B^2}$. To get the second root, it is evident DE′ must be subtracted from DF. Laying off from D to E″, in *a contrary direction*, DE″ = DE′, FE″ will be the root, and will be equal to $C + A + \sqrt{A^2 + B^2}$, and this value will be positive, if the subtraction is possible; that is if C or its equal DF is greater than DE′, and negative, if less.

23. In general, when a negative sign is attached to a re-

sult in Algebra, it is always the index of subtraction. If the expression contain positive quantities, on which this subtraction can be performed, the indication of the sign is satisfied. If not, the sign remains, to *indicate* the operation yet to be performed. To interpret the result in this case, we must conceive a more general question, which contains quantities, on which the indicated operation may be performed, and discover the signification to be given to the result.

EXAMPLES.

1. Construct $\dfrac{abc + def - ghi}{l\,m}$.

2. Construct $\sqrt{a}$.

3. Construct $\sqrt{a^2 + b^2 + c^2 + d^2}$.

# CHAPTER II.

## DETERMINATE GEOMETRY.

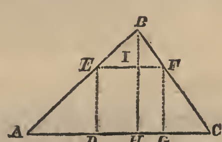
24. ANALYTICAL GEOMETRY is divided into two parts:

1st. *Determinate Geometry*, which consists in the application of Algebra to determinate problems, that is, to problems which admit of only a finite number of solutions.

2ndly. *Indeterminate Geometry*, which consists in the investigation of the general properties of lines, surfaces and solids, by means of analysis.

25. We will first apply the principles explained in the first chapter, to the resolution and construction of problems of Determinate Geometry.

Having given the base and altitude of a triangle, it is required to find the side of the inscribed square. Let ABC be the proposed triangle, of which AC is the base, and BH the altitude. Designate the base by $b$, and the altitude by $h$, and let $x$ be the side of the inscribed square. The side EF, being parallel to AC, the triangles BEF and ABC are similar; and we have,




$$AC : BH :: EF : BI,$$

or

$$b : h :: x : h - x.$$

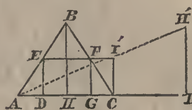
Multiplying the means and the extremes together, and putting the products equal to each other, we have,

$$bh - bx = hx$$

$$x = \frac{bh}{b+h}$$

from which the numerical value of $x$ may be determined, when $b$ and $h$ are known.

26. We may also from this expression, find the value of $x$ by a geometrical construction, since it is evidently the fourth proportional to the lines $b+h$, $b$, and $h$. Produce AC to B′, making CB′ $= h$, erect the perpendicular B′H′ $= h$, join A and H′, and through C draw CI′ parallel to H′ B′, it will be the side of the required square, and drawing through I′ a parallel to the base, DEFG will be the inscribed square. For, the triangles AB′H′, ACI′ being similar, we have,




$$AB' : B'H' :: AC : CI'$$

or

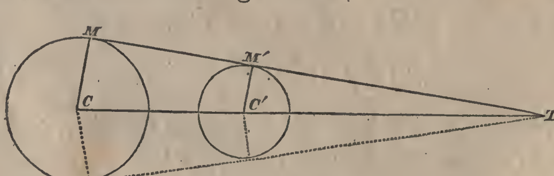
$$b+h : h :: b : x;$$

hence

$$x = \frac{bh}{b+h}.$$

27. There are some questions of a more complicated nature, than the one which we have just considered, but which when applied to analysis lead to the most simple and satisfactory results.

Let it be required to draw a common tangent to two circles, situated in the same plane, their radii and the distance between their centres being known.




Let us suppose the problem solved, and let MM′ be the common tangent. Produce MM′ until it meets the straight line joining the centres at T. The angles CMT and C′M′ T

being right, the triangles CMT and C′M′ T will be similar, and give the proportion,

$$CM : C'M' :: CT : C'T.$$

Designating the radii of the two circles by $r$ and $r'$, the distance between the centres by $a$, and the distance CT by $x$, the above proportion becomes,

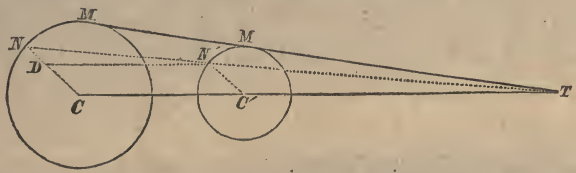
$$r : r' :: x : x - a,$$

or
$$rx - ra = r'x;$$

hence
$$x = \frac{ar}{r - r'},$$

which shows that the distance $CT = x$ is a fourth proportional to the three lines $r - r'$, $a$, and $r$.

*To draw the tangent line.*




Through the centres C and C′, draw any two parallel radii CN, C′N′, the line NN′ joining their extremities will cut the line joining the centres, at the same point T, from which, if a tangent be drawn to one circle, it will be tangent to the other also. For, the triangles CNT, C′N′T, will still be similar, since the angles at N and N′ are equal, and will give the same proportion. But to show the agreement of this construction with the algebraic expression for $x$, draw through N′, N′D parallel to CC′, N′D will be equal to $a$, and ND to $r - r'$; the triangles N′DN, CNT, being similar, give the proportion,

$$ND : DN' :: NC : CT,$$

or $$r - r' : a :: r : \mathrm{CT};$$

hence $$\mathrm{CT} = \frac{ar}{r - r'},$$

which is the same value found before. TMM′ drawn tangent to one circle, will also be tangent to the other. As two tangents can be drawn from the point T, the question admits of two solutions.

28. If we suppose, in this example, the radius $r$ of the large circle to remain constant, as well as the distance between the centres, the product $ar$ will be constant. Let the radius $r'$ of the small circle increase, as $r'$ increases, the denominator $r - r'$ will continually diminish, and will become zero, when $r = r'$. The value of $x$ then becomes $\frac{ar}{o} =$ infinity. This appears also from the geometrical construction, for when the radii are equal, the tangent and the line joining the centres are parallel, and of course can only meet at an infinite distance.

29. If $r'$ continue to increase, the denominator becomes negative, and since the numerator is positive, the value of $x$ will no longer be infinite, but negative, and equal to $-$ CT, which shows that the point T is changed in position (Art. 9), and is now found on the left of the circle whose radius is $r$.

30. To construct a rectangle, when its surface and the difference between its adjacent sides are given:

Let $x$ be the greater side, $2a$ the difference, $x - 2a$ will be the less. Let $b$ be the side of the square, whose surface is equal to that of the rectangle. This condition will give,

$$x\,(x - a) = b^2 \text{ or } x^2 - 2ax = b^2\,;$$

from which we obtain the two values,

$$x = a + \sqrt{a^2 + b^2},\ x = a\ \sqrt{a^2 + b^2}.$$

There are the same values of $x$ constructed in Art. 18, the first being represented by DE, the second by — DE. But we can easily verify this, and show that DE $= a + \sqrt{a^2 + b^2}$ is the greater side of the rectangle. For, if we subtract from this value the difference $2a$, the remainder $-a + \sqrt{a^2 + b^2}$ multiplied by the greater side, is equal to $b^2$, the surface of the rectangle, $-a + \sqrt{a^2 + b^2}$ is therefore the smaller side.

31. We see also that the second value of $x$ taken with a contrary sign, represents the smaller side of the rectangle. Hence the calculation not only gives us the greater side, which alone was introduced as the unknown quantity, but also the less. This arises from the general nature of all algebraic results, by virtue of which the equation which expresses the conditions of the problem, gives, at the same time, every value of the unknown quantity which will satisfy these conditions. In the example before us we have represented the greater side by $+ x$, and have found that its value depended upon the equation

$$x^2 - 2ax = b^2.$$

If we had made the smaller side the unknown quantity, and represented its value by $- x$, which we were at liberty to do, it would have depended upon the equation

$$-x\,(-x + 2a) = b^2, \text{ or } x^2 - 2ax = b^2,$$

which is the same equation as the preceding. Hence, this equation should not only give us the greater side, which was at first represented by $+ x$, but also the less, which in this instance is represented by $- x$.

32. The preceding examples are sufficient to indicate generally the steps to be taken, to express analytically the conditions of geometrical problems:

1st. We commence by drawing a figure, which shall represent the several parts of the problem, and then such other lines, as may from the nature of the problem lead to its solution.

2d. Represent, as in Algebra, the known and unknown parts by the letters of the alphabet.

3d. Express the relations which connect these parts by means of equations, and form in this manner as many equations as unknown quantities; the resolution of these equations will determine the unknown quantities, and resolve the problem proposed.

### EXAMPLES.

1. In a right-angled triangle, having given the base, and the difference between the hypothenuse and perpendicular; find the sides.

2. Having given the area of a rectangle, inscribed in a given triangle; determine the sides of the rectangle.

3. Determine a right-angled triangle; having given the perimeter and the radius of the inscribed circle.

4. Having given the three sides of a triangle; find the radius of the inscribed circle.

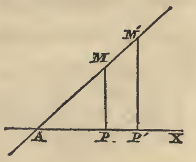
5. Determine a right-angled triangle, having given the hypothenuse and the radius of the inscribed circle.

6. Determine the radii of the three equal circles, described in a given circle, which shall be tangent to each other, and also to the circumference of the given circle.

# CHAPTER III.

## INDETERMINATE GEOMETRY.

33. In the questions which we have been considering, the conditions have limited the values of the required parts. We propose now to discuss some questions of Indeterminate Geometry, which admit of an infinite number of solutions.

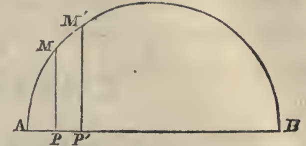



For example, let us consider any line AMM'. From the points M, M', let fall the perpendiculars MP, MP', upon the line AX taken in the same plane. These perpendiculars will have a determinate length, which will depend upon the nature and position of the line AMM', and the distance between the points M, M', &c. Assuming any point A on the line AX, each length AP will have its corresponding perpendicular MP, and the relation which subsists between AP, PM; AP', P'M'; for the different points of the line AMM' will necessarily determine this line. Now, this relation may be such as to be always expressed by an equation, from which the values of AP, AP', &c. can be found, when those of PM, P'M', are known. For example, suppose AP = PM, AP' = P'M', &c. representing the bases of these triangles by $x$, and the perpendiculars by $y$, we have the relation

$$y = x.$$

In this case, the series of points MM', &c., forms evidently the straight line AMM', making an angle of 45° with AX.

34. Again, suppose that the condition established was




such, that each of the lines PM, P′M′, should be a mean proportional between the distances of the points P, P′, &c. from the points A and B taken on the line AB. Calling PM $y$, AP $x$, and the distance AB $2a$, we would have,

$$y^2 = x\,(2a - x), \quad \text{or, } y^2 = 2ax - x^2.$$

This equation enables us to determine $y$ when $x$ is known, and reciprocally, knowing the different values of $x$, we can determine those of $y$. It is evident that this line is the circumference of a circle described on AB as a diameter.

35. Since each of the equations

$$y = x, \qquad y^2 = 2ax - x^2,$$

serves to determine all the points of the straight line and circle, it follows that they are equivalent to the actual construction of these lines, and may be used to represent them.

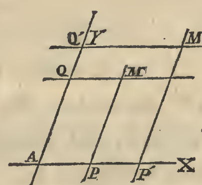
36. Generalizing this result, we may regard every line as susceptible of being represented by an equation between two indeterminate variables; and, reciprocally, every equation between two indeterminates may be interpreted geometrically, and considered as representing a line, the different points of which it enables us to determine. It is this more extended application of Algebra to Geometry, that constitutes the *Science of Analytical Geometry.*

### *Of Points, and the Right Line in a Plane.*

37. As all geometrical investigations refer to the positions of points, our first step must be to show how these positions are expressed and fixed by means of analysis.

38. *Space* is indefinite extension, in which we conceive all bodies to be situated. The *absolute* positions of bodies cannot be determined, but their *relative* positions may be, by referring them to objects whose positions we suppose to be known.

39. The *relative* positions of all the points of a plane are determined by referring them to two straight lines, taken at pleasure, in that plane, and making any angle with each other.




Let AX and AY be these two lines, every point M situated in the plane of these lines, is known, when we know its distances from the lines AX and AY measured on the parallels PM and QM to these lines, respectively.

The lines QM, Q'M', or their equals AP, AP', are called *abscissas*, and the lines PM, P'M', or their equals AQ, AQ', *ordinates*. The line AX is called the *axis of abscissas*, or simply the *axis of x's*, and the line AY the *axis of ordinates*, or the *axis of y's*. The ordinates and abscissas are designated by the general term *co-ordinates*. AX and AY are then the *co-ordinate axes*, and their intersection A is called the *origin of co-ordinates*.

40. It may be proper here to remark, that the terms *line* and *plane* are used in their most extensive signification,—that is, they are supposed to extend indefinitely in both directions.

41. Let us represent the abscissas by $x$, and the ordinates by $y$, $x$ and $y$ will be *variables*, which will have different values for the different points which are considered. If, for example, having measured the lengths AP, PM, which determine the point M, we find the first equal to $a$, and the

second equal to $b$, we shall have for the equations which fix this point,

$$x = a, \qquad y = b.$$

These are called *the equations of the point* M.

42. If the abscissa AP remain constant, while the ordinate PM diminishes, the point M will continually approach the axis AX; and when PM $= o$, the point M will be on this axis, and its equations become

$$x = a, \qquad y = o.$$

If the ordinate PM remain constant, while the abscissa AP diminishes, the point M will continually approach the axis AY, and will coincide with it when AP $= o$; the equations will then be,

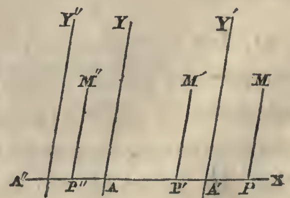
$$x = o, \qquad y = b.$$

Finally, if AP and PM become zero at the same time, the point M will coincide with the point A, and we have,

$$x = o, \qquad y = o,$$

for the equations of the origin of co-ordinates.

43. From this discussion we see, that, in giving to the variables $x$ and $y$ every possible *positive* value, from zero to infinity, we may express the position of every point in the angle YAX. But how may points situated in the other angles of the co-ordinate axes be expressed?




Instead of taking YA for the axis of $y$, take another line, Y'A', parallel to YA and in the same plane, at a distance AA' $=$ A from the old axis.

Calling $x'$ the new abscissas,

counted from the origin A′, we have for the point M, situated in the angle Y′A′X,

$$AP = AA' + A'P,$$

or, $$x = A + x'.$$

But if we consider a point M′ in the angle Y′A′A, we have,

$$AP' = AA' - A'P,$$

or, $$x = A - x'.$$

Hence, in order that the same analytical expression,

$$x = A + x',$$

may be applicable to points situated in both these angles, we must regard the values of $x'$ as negative for the angle AA′Y′, so that the change of sign corresponds to the change of position with respect to the axis A′Y′.

44. To confirm this consequence, and show more clearly how the preceding formula can connect the different points in these different angles, let us consider a point on the axis A′Y′. For this point we have $x' = o$, and the formula

$$x = A + x'$$

becomes $$x = +A.$$

This is the value of the abscissa AA′ with respect to AX, AY. But if we wish that this equation suit points on the axis AY, for any point of this axis $x = o$, and the preceding formula will give,

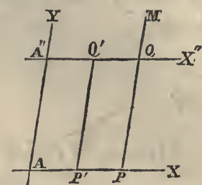
$$x' = -A,$$

which is the same value of the abscissa AA′ referred to the axis A′Y′. The analytical expression for this abscissa becomes then positive for the axis AY, and negative for the axis A′Y′, when we consider the different points of the plane connected by the equation

$$x = A + x'.$$

This result applies equally to the negative values of $x$, and proves that they belong to points situated on the opposite side of the axis AY to the positive values.

45. Moving the axis AX parallel to itself, and fixing the new origin at A″, making AA″ = B, and calling $y'$ the new ordinates counted from A″, we have,




$$y = B + y'$$

for points in the angle YA″X″, and

$$y = B - y'$$

for those in the angle AA″X″. To express points situated in both these angles by the same formula, we must regard those points corresponding to negative values of $y'$ as lying on the opposite side of the axes of A″X″ to the positive values; and as this applies equally to the axes AX and AY, we conclude that the change of sign in the variable $y$ corresponds to the change of position of points with respect to the axis of abscissas.

46. From what has been said, we conclude, that if the abscissas of points lying on the right of the axis of $y$ be *assumed* as positive, those of points lying on the left of this axis will be negative; and also if the ordinates of points lying above the axis of $x$ be assumed as positive, those below this axis will be negative. We shall have, therefore,

In the first angle, $x$ positive and $y$ positive;

In the second angle, $x$ negative and $y$ positive;

In the third angle, $x$ positive and $y$ negative;

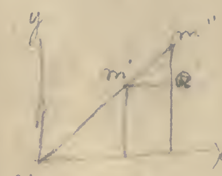
In the fourth angle, $x$ negative and $y$ negative;

and the equations

$$x = a, \qquad y = b,$$

which determine the position of a point in the angle YAX, become, successively,

$$x = -a, \qquad y = +b;$$
$$x = +a, \qquad y = -b;$$
$$x = +a, \qquad y = -b.$$




47. From what precedes we may find the analytical expression for the distance between two points, when we know their co-ordinates referred to rectangular axes. Let M', M'', be the given points. Draw M'Q' parallel to the axis of $x$. The right-angled triangle M'M''Q' gives,

$$\mathrm{M'M''} = \sqrt{\mathrm{M'Q'}^2 + \mathrm{M''Q'}^2}$$

Let $x'$ $y'$ represent the co-ordinates of the point M', $x''$ $y''$ the co-ordinates of the point M''; $\mathrm{M'Q'} = x'' - x'$, and $\mathrm{M''Q'} = y'' - y'$; representing the distance between the two points by D, we have,

$$\mathrm{D} = \sqrt{(x'' - x')^2 + (y'' - y')^2}.$$

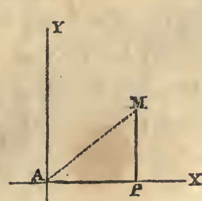
If the point M' were placed at the origin A, its co-ordinates would become

$$x' = o, \qquad y' = o,$$

and the value of D would reduce to

$$\mathrm{D} = \sqrt{x''^2 + y''^2}.$$

Which is the expression for the distance of a point from the origin of co-ordinates. This value is easily verified, for the triangle AMP being right-angled, gives,

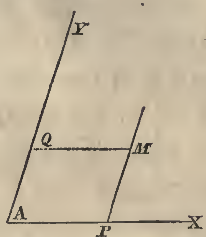


$$\overline{AM}^2 = \overline{AP}^2 + \overline{PM}^2 = x^2 + y^2$$

or

$$D = \sqrt{x^2 + y^2}.$$

48. Let us resume the equations $x = a$, $y = b$, which determine the positions of a point in a plane, $a$ and $b$ being any quantities whatever.




The equation $x = a$ considered by itself, corresponds to every point whose abscissa is equal to $a$. Take $AP = a$. Every point of the line PM drawn parallel to AY, and extending indefinitely in both directions, will satisfy this condition. $x = a$ is therefore the equation of a line drawn parallel to the axis of $y$, and at a distance from this axis equal to $a$. In like manner $y = b$ is the equation of a straight line parallel to the axis of $x$. The point M, which is determined by the equations

$$x = a, \qquad y = b,$$

is therefore found at the intersection of two straight lines drawn parallel to the co-ordinate axes. The line whose equation is $x = a$ will be on the positive side of the axis of $y$ if $a$ is positive, and the reverse if $a$ is negative. If $a = o$, it will coincide with the axis of $y$, and the equation of this axis will be

$$x = o.$$

The straight line whose equation is $y = b$ will be situated above or below the axis of $x$, according as $y$ is positive or negative. When $y = o$, it will coincide with the axis of $x$, and the equation of this axis is therefore

$$y = o.$$

Finally, the origin of co-ordinates being at the same time on the two axes, will be defined by the equations

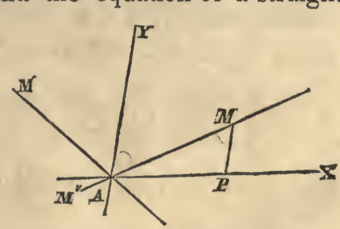
$$x = o, \quad y = o,$$

as we have before found.

49. The method which we have used to express analytically the position of a point, may be therefore used to designate a series of points, situated on the same straight line parallel to either at the co-ordinate axes. Generalizing this result, we see, that if there exist the same relation between the co-ordinates of all the points of any line whatever, the equation in $x$ and $y$ which expresses this relation, must characterize the line. Reciprocally, the equation being given, the nature of the line is determined, since for every value of $x$ or $y$ we may find the corresponding value of the other co-ordinate.

50. *An equation which expresses the relation which exists between the co-ordinates of every point of a line, is called the equation of that line.*

51. Let it be required to find the equation of a straight line passing through the origin of co-ordinates, and making an angle $\alpha$ with the axis of $x$. Let the angle which the co-ordinate axes make with each other be called $\beta$. From any point M draw PM parallel to the axis of $y$, we will have,




$$\frac{\text{PM}}{\text{AP}} = \frac{\sin\ \alpha}{\sin\ (\beta - \alpha)}, \text{ or } y = x\ \frac{\sin\ \alpha}{\sin\ (\beta - \alpha)}$$

sin α : sin (β − α) :: x : y

sin α × AP = sin (β − α) P

and as this equation exists for every point of the line AM, it is the equation of that line.

52. The value of $\alpha$ is the same for every point of the line AM, but varies from one line to another. If we suppose $\alpha$ to diminish, the line AM will incline more and more to the axis of $x$, and when $\alpha = o$ coincides with this axis. In this case the analytical expression becomes $y = o$, which is the same equation for the axis of $x$ which was found before.

53. Again, let $\alpha$ increase. The line AM approaches the axis AY and coincides with it when $\alpha = \beta$. In this case the $\sin(\beta - \alpha) = o$, and the equation becomes $x = o$, which is the equation of the axis of $y$.

54. If $\alpha$ continue to increase $(\beta - \alpha)$ become negative, and the equation becomes

$$y = -x \frac{\sin \alpha}{\sin(\beta - \alpha)}$$

and is the equation of the line AM'. When $\alpha = 180°$, $\sin \alpha = o$, and the line coincides with the axis of $x$, and we have again $y = o$.

55. Finally, for $\alpha > 180°$ $\sin \alpha$ is negative, as well as $\sin(\beta - \alpha)$, and the equation becomes

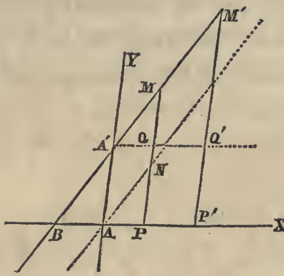
$$y = x \frac{\sin \alpha}{\sin(\beta - \alpha)}$$

and represents the line MAM''. Hence the formula

$$y = x \frac{\sin \alpha}{\sin(\beta - \alpha)}$$

is applicable to every straight line drawn through the origin of co-ordinates.

56. Let us now consider a line A'M' making the same




angle $\alpha$ with the axis of $x$, but which does not pass through the origin; and as its inclination to the axis of $x$ does not determine its position, suppose it cut the axis of $y$ at a distance AA' from the origin, equal to $b$. The equation of a line parallel to A' M', and passing through the origin, will be

$$y = x \frac{\sin \alpha}{\sin (\beta - \alpha)}$$

The value of any ordinate PM will be composed of the part PN $= x \dfrac{\sin \alpha}{\sin (\beta - \alpha)}$ and MN $=$ AA' $= b$. Hence

$$y = x \frac{\sin \alpha}{\sin (\beta - \alpha)} + b;$$

which is the most general equation of a straight line considered in a plane.

57. To find the point in which this line cuts the axis of $x$, make $y = o$, which is the condition for every point of this axis; and making $x = o$, determines the point in which it cuts the axis of $y$.

Should the line A' M' cut the axis of $y$ below the origin of co-ordinates, the value of the new ordinate would be less than that of the ordinate of the line passing through the origin, by the distance cut off on the axis of $y$; hence we have for the equation of the line,

$$y = x \frac{\sin \alpha}{\sin (\beta - \alpha)} - b.$$

58. In this discussion we have supposed the co-ordinate axes to make any angle $\beta$ with each other. They are most generally taken at right-angles, since it simplifies the calculation. If therefore $\beta = 90$

$$\sin(\beta - \alpha) = \sin(90^\circ - \alpha) = \cos\alpha,$$

and the equation becomes

$$y = x\frac{\sin\alpha}{\cos\alpha} + b = x\tan\alpha + b.$$

Representing the tangent of $\alpha$ by $a$, the equation is

$$y = ax + b,$$

which is the equation of a right line referred to rectangular axes. In this equation $a$ represents the tangent of the angle which the line makes with the axis of $x$, and $b$ the distance from the origin at which it cuts the axis of $y$.

59. The most general form of an equation of the first degree is

$$\mathrm{A}y + \mathrm{B}x + \mathrm{C} = o,$$

which may also be written thus

$$y = -\frac{\mathrm{B}}{\mathrm{A}}x - \frac{\mathrm{C}}{\mathrm{A}}$$

This equation will be of the same form as that just discussed, if we make

$$a = -\frac{\mathrm{B}}{\mathrm{A}} \text{ and } b = -\frac{\mathrm{C}}{\mathrm{A}}$$

*Hence every equation of the first degree between two variables is the equation of a straight line.*

Every equation of this form, whatever be the number of variables, is called a *linear equation.*

60. So long as $a$ and $b$ are indeterminate, the position of

the line is unknown. The equation only shows that its points are on a straight line. But if $a$ and $b$ be known, the position of the line is fixed, since we know one of its points on the axis of $y$, and the angle it makes with the axis of $x$.

The determination of the co-efficients $a$ and $b$ leads to some interesting questions, which we will now examine.

61. To find the equation of a straight line, which shall pass through two given points:

Let $x'$ $y'$, $x''$ $y''$, be the co-ordinates of these points. The line being straight, its equation will be of the form

$$y = ax + b;$$

it is required to determine $a$ and $b$.

Since the required line must pass through the point whose co-ordinates are $x'$, $y'$, its equation must be satisfied when $x'$ and $y'$ are substituted for $x$ and $y$; hence

$$y' = ax' + b.$$

But it also passes through the point whose co-ordinates are $x''$ $y''$. We have for the same reason,

$$y'' = ax'' + b.$$

These two equations determine $a$ and $b$. Substituting their values in the given equation, the line will be determined. The elimination is very easily performed, by subtracting the second equation from the first, and the third from the second, which give

$$y - y' = a\,(x - x')$$
$$y' - y'' = a\,(x' - x'');$$

from which we have

$$y - y' = \frac{y' - y''}{x' - x''}\,(x - x'), \text{ and } a = \frac{y' - y''}{x' - x''}.$$

The first of these equations is that of the required line, and the second determines the angle which it makes with the axis of $x$. It is easy to show that the conditions of the problem are fulfilled; for $x = x'$ gives $y = y'$, and $x = x''$ gives $y = y''$. If $y' - y'' = o$, we have $a = o$ and $y = y''$, which shows that the line is parallel to the axis of $x$. If $x' - x'' = o$, we have $\frac{1}{a} = o$ and $x = x''$, which shows that the line is perpendicular to the axis of $x$.

62. To find the conditions necessary that a straight line be parallel to a given straight line.

Let

$$y = ax + b$$

be the equation of the given line, in which $a$ and $b$ are known. That of the required line will be of the form

$$y = a'x + b',$$

in which $a'$ and $b'$ are unknown.

In order that these lines should be parallel, it is necessary that they should make the same angle with the axis of $x$. Hence

$$a = a',$$

and the equation of the parallel, after substitution, becomes

$$y = ax + b',$$

in which $b'$ is *indeterminate*, since an infinite number of lines may be drawn parallel to a given line.

63. Were it required that the line should pass through a point whose co-ordinates are $x'$, $y'$, they must satisfy the equation, and we have

$$y' = ax' + b'.$$

These two equations determine $b'$, and, combining them, we have

$$y - y' = a(x - x')$$

for the equation of a line drawn through a given point parallel to a given line.

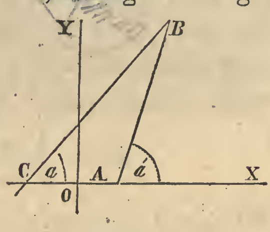
64. To find the angle included between two lines, given by their equations.

Let

$y = ax + b$ be the equation of the first line,

$y = a'x + b'$ the equation of the second line.

The first line makes with the axis of $x$, an angle the trigonometrical tangent of which is $a$; the second, an angle whose tangent is $a'$. The angle sought is ABC $= \alpha' - \alpha$, since BAX $=$ ACB $+$ CBA. But we have from Trigonometry,




$$\text{tang}\ (\alpha' - \alpha) = \frac{\text{tang}\ \alpha' - \text{tang}\ \alpha}{1 + \text{tang}\ \alpha'\ \text{tang}\ \alpha}.$$

Calling ABC $=$ V, and putting for tang $\alpha$ and tang $\alpha'$ $a$ and $a'$, we have

$$\text{tang V} = \frac{a' - a}{1 + aa'}.$$

65. If the lines be parallel, V $= o$; and the tang V $= o$, which gives $a - a' = o$ and $a = a'$, which agrees with the condition before established, (Art. 62.)

66. If the lines be perpendicular to each other, V $= 90°$ and

$$\text{tang V} = \frac{a' - a}{1 + aa'} = \alpha,$$

which gives

$$1 + aa = o,$$

which is the condition that two straight lines should be perpendicular to each other. If one of the quantities $a$ or $a'$ be known, the other is determined by this equation.

67. To find the intersection of two straight lines, given by their equations.

Let

$$y = ax + b,$$

$$y = ax + b',$$

be the equations of the two lines. As the point of intersection is on both of the lines, its co-ordinates must satisfy at the same time the two equations. Combining them, we shall deduce the values of $x$ and $y$ which correspond to the point of intersection. We have by elimination,

$$x = -\frac{b - b'}{a - a'}, \qquad y = \frac{ab' - a'b}{a - a}.$$

When $a = a'$, these values become infinite. The lines are then parallel, and can only intersect at an infinite distance.

68. The method which we have just employed is general, and may be used to determine the points of intersection of two curve lines, situated in the same plane, when we know their equations; for, as these points must be at the same time on both curves, their co-ordinates must satisfy the equations of the curves. Hence, combining these equations, the values we deduce for $x$ and $y$ will be the co-ordinates of the points of intersection.

### EXAMPLES.

1. Construct the line whose equation is

$$y = 3x + 5.$$

2. Construct the line whose equation is

$$y = x + 1.$$

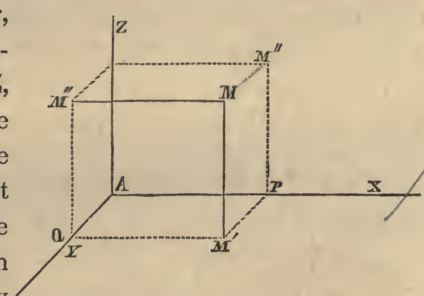
3. Construct the line whose equation is

$$y = -x - 2.$$

4. Draw from a given point a line perpendicular to a given straight line, and find the length of the perpendicular.

### *Of Points, and the Straight Line in Space.*

69. A point is determined in space, when we know the length and direction of three lines, drawn through the point, parallel to three planes, and terminated by them.




For more simplicity we will suppose three planes at right angles to each other, and let them be represented by YAX, XAZ, ZAY. Suppose the point M at a distance MM′, from the first plane, MM″ from the second, and MM‴ from the third. If we draw through these lines, three planes parallel to the rectangular planes, their intersection will give the point M. The rectangular planes to which points in space are referred, are call *Co-ordinate Planes.* They intersect each other in the lines AX, AY, AZ, passing through the point A and perpendicular to each other. The distance MM′ of the point M from the plane YAX, may be laid off on the line AZ, and is equal to AR. Likewise the distance MM″ may be laid off on AY, and is AQ. Finally, AP laid off on AX is equal to MM‴.

70. The lines AX, AY, AZ, on which hereafter the respective distances of points from the co-ordinate planes will be reckoned, are called the *Co-ordinate Axes*, and the point A is the *Origin*.

71. Let us represent by $x$ the distances laid off on the first, which will be the axis of $x$, by $y$ those laid off on A$y$, which will be the axis of $y$, and by $z$ those laid off on AZ, which will be the axis of $z$.

If then the distances AP, AQ, AR, be measured and found equal to $a$, $b$, $c$, we shall have to determine the point M, the three equations

$$x = a, \quad y = b, \quad z = c.$$

These are called *the Equations of the point* M.

72. The points M′, M″, M‴, in which the perpendiculars from the point M meet the co-ordinate planes, are called the *Projections of the point* M.

These projections are determined from the three equations given above, for we obtain from them

| | | |
|---|---|---|
| $y = b,\ x = a,$ | which are the equations | of the projection M′, |
| $x = a,\ z = c,$ | " " " | of the projection M″, |
| $z = c,\ y = b,$ | " " " | of the projection M‴; |

and we see from the composition of these equations, that two projections being given, the other follows necessarily.

In the geometrical construction they may be easily deduced from each other. For example, M″, M‴, being given, draw M‴Q, M″P, parallel to AZ, and QM′, PM′, parallel respectively to AX and AY, M′ will be the third projection of the point M.

73. There results from what has been said, that all points in space being referred to three rectangular planes, the points

in each of these planes are naturally referred to the two perpendiculars, which are the intersections of this plane with the other two.

74. The plane YAX is called the plane of $x$'s, and $y$'s, or simply $xy$;

The plane XAZ, that of $x$'s, and $z$'s, or $xz$;

And the plane ZAY, that of $z$'s, and $y$'s, or $zy$.

75. The same interpretation is given to negative ordinates, as we have before explained, and the signs of the co-ordinates $x, y, z$, will make known the positions of points in the four angles of the co-ordinate planes.

76. Let us resume the equations,

$$x = a, \quad y = b, \quad z = c;$$

$a, b, c$, being indeterminate.

The first $x = a$ considered by itself, belongs to every point whose abscissa AP is equal to $a$. It belongs therefore to the plane MM$'$PM$''$, supposed indefinitely extended in both directions. For every point of this plane, as it is parallel to the plane ZAY, satisfies this condition. The equation $y = b$ corresponds to every point of the plane MM$'''$ QM$'$, drawn through the point M parallel to ZAX, and finally $z = c$ corresponds to every point of the plane MM$''$ RM$'''$ drawn through M parallel to the plane XAY. Hence the equations

$$x = a, \quad y = b, \quad z = c,$$

show that the point M is situated at the same time on three planes drawn parallel respectively to the co-ordinate planes, and at distances represented by $a, b, c$.

77. When these distances are nothing, the equations become

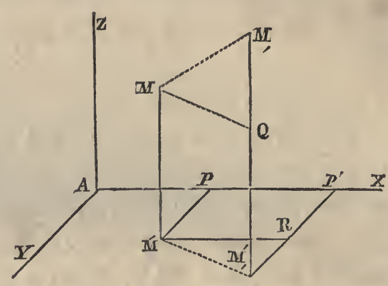
$$x = o, \quad y = o, \quad z = o,$$

which are the equations of the origin. The first of these $x = o$ corresponds to the plane $yz$, the second $y = o$ to the plane $xz$, and the third $z = o$ to the plane $xy$. Since for every point of these planes, these separate conditions exist.

78. To find the expression for the distance between two points whose co-ordinates are known, let M $M_{\prime}$ be the two points, whose co-ordinates are $x$, $y$, $z$ ; $x'$, $y'$, $z'$ ; if through the first we draw a line MQ parallel to the plane of $xy$, and terminated by the ordinate $M_{\prime}$ $M'_{\prime}$ we shall have

$$\overline{MM_{\prime}}^2 = \overline{QM}^2 + \overline{QM_{\prime}}^2.$$

$QM_{\prime}$ is equal to $M'_{\prime}$ $M_{\prime}$ — $M'M$ or $z - z'$; QM is equal to $M'M'_{\prime}$. If, through the point $M_{\prime}'$ we draw $M'R$ parallel to the axis of $x$, $M'_{\prime}$ R = $y' - y$; $M'R = x' - x$, and we shall have,




$$\overline{M'M'_{\prime}}^2 = \overline{M'R}^2 + \overline{M'_{\prime}R}^2 = (y' - y)^2 + (x' - x)^2.$$

Substituting these values, we have,

$$\overline{MM_{\prime}}^2 = (z' - z)^2 + \overline{M'M'_{\prime}}^2 = (z' - z)^2 + (y' - y)^2 + (x' - x)^2.$$

Calling D the required distance, this value becomes

$$D = \sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}$$

which is the expression for the distance between any two points in space.

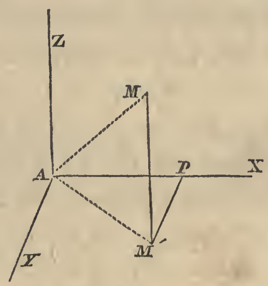
79. We may remark that $x' - x$, $y' - y$, $z' - z$, are the projections of the line D on the three axes of $x$, $y$, $z$, from

which results this theorem. *The square of any portion of a straight line is equal to the sum of the squares of its projections on the three rectangular axes.*

80. If one of the points, as that whose co-ordinates are $x$, $y$, $z$, coincide with the origin A, the preceding formula would become,

$$D = \sqrt{x'^2 + y'^2 + z'^2},$$

which expresses the distance of a point in space from the origin of the co-ordinates. In fact, the triangles MAM′, AM′P, being right-angled at M′ and P, give,




$$\overline{AM}^2 = \overline{MM'}^2 + \overline{AM'}^2 = \overline{MM'}^2 + \overline{M'P}^2 + \overline{AP}^2 = z^2 + y^2 + x^2,$$

as we have just found.

We see by this result, that *the square of the diagonal of a rectangular parallelopipedon is equal to the sums of the squares of its three edges.*

81. This last result gives a relation between the cosines of the angles which any line AM makes with the co-ordinate axes. For, let these angles be represented by X, Y, Z, calling $r$ the distance AM. In the right-angle triangle AMM′ we have MM′ equal to $z$, and AMM′ = MAZ in the angle Z, and we have,

$$z = r \cos Z.$$

Reasoning in the same manner, the others give,

$$y = r \cos Y,$$

$$x = r \cos X.$$

Squaring these three equations and adding them together, we get,

$$x^2+y^2+z^2=r^2(\cos^2 X+\cos^2 Y+\cos^2 Z);$$

but,

$$x^2+y^2+z^2=r^2.$$

Hence,

$$r^2=r^2(\cos^2 X+\cos^2 Y+\cos^2 Z),$$

$$1=\cos^2 X+\cos^2 Y+\cos^2 Z,$$

*which proves that the sum of the squares of the cosines of the angles which a straight line in space makes with the co-ordinate axes, is always equal to unity.*

82. Let us now determine the equation of a straight line. To do this, we will remark, that if several points be in a straight line in space, their projections on the co-ordinates' planes will also be in straight lines; for, the projection of a point on a plane is the foot of a perpendicular let fall from the point on this plane. The straight line which contains the several points will be in the same plane with the perpendiculars drawn through these points, and consequently the points in which these perpendiculars meet the co-ordinates' plane will be in one and the same straight line. The plane which contains these perpendiculars is called the *Projecting Plane of the Line*, and its intersection with the co-ordinate plane the *Projection of the Line.*

83. A straight line is determined when we know two planes which contain it: it will therefore be known when we have two of its projecting planes, and these are determined when we know the projections through which they pass. Hence, a straight line is determined when we know its projections on two of the co-ordinate planes. And as the equations of these projections on the planes of $x\,z$ and $y\,z$ are

$$x=a\,z+\alpha, \qquad y=b\,z+\beta,$$

these equations fix the position of the line in space.

If the line pass through the origin, $\alpha = o$, and $\beta = o$.

84. These results are easily verified; for the equation

$$x = az + \alpha$$

being independent of $y$, is not only the equation of the projection of the given line on the plane of $xz$, but corresponds to every point of the projecting plane of the given line, of which this projection is the trace. It is therefore the equation of this plane.

Likewise the equation

$$y = bz + \beta$$

being independent of $x$, not only represents the equation of the projection of the given line on the plane of $yz$, but is the equation of the plane which projects this line on the plane of $yz$. Consequently the system of equations

$$x = az + \alpha, \qquad y = bz + \beta,$$

signifies that the given line is situated at the same time on both these planes. Hence they determine its position.

85. Eliminating $z$ from these equations, we get,

$$\frac{x - \alpha}{a} = \frac{y - \beta}{b}, \text{ or } y - \beta = \frac{b}{a}(x - \alpha),$$

which is the equation of the projection of the given line on the plane of $yx$, and also corresponds to the plane which projects this line on the plane of $xy$.

86. We conclude from these remarks, that, in general, two equations are necessary to fix the position of a line in space, and these equations are those of the two planes, whose intersection determines the line. When a line is situated in one of the co-ordinate planes, its projections on

the other two are in the axes. If, for example, it be in the plane of $xz$, we have for this plane,

$$b = o, \qquad \beta = o;$$

and its equations become

$$y = o, \qquad x = az + \alpha.$$

The first shows that the projection of the line on the plane of $yz$ is in the axis, and the second is the equation of its projection on the plane of $xz$, which is the same as for the line itself, with which it coincides.

87. Let us resume the equations

$$x = az + \alpha, \qquad y = bz + \beta.$$

So long as the quantities, $a$, $b$, $\alpha$, $\beta$, are unknown, the position of the line is undetermined. If one of them, $a$ for example, be known, this condition requires that the line shall have such a position in space, that its projection on the plane of $xz$, shall make an angle with the axis of $z$, the tangent of which is $a$. If $\alpha$ be also known, this projection must cut the axis of $x$ at this given distance from the origin, and these two conditions will limit the line to a given plane.

If $b$ be known, a similar condition will be required with respect to the angle which its projection on the plane of $yz$ makes with the axis of $z$; and finally, if all four constants be known, the line is completely determined.

88. The determination of the constants $a$, $b$, $\alpha$, $\beta$, from given conditions, and the combination of the lines which result from them, lead to questions which are analogous to those we have been considering.

89. Before proceeding to their discussion, we will remark, that the methods which we have just used, may be applied

to curve as well as straight lines. In fact, if we know the equations of the projections of a curve on two of the co-ordinate planes, we can for every value of one of the variables $x$, $y$ or $z$, find the corresponding values of the other two, which will determine points on the curve in space.

90. The projection of a curve is the intersection of a cylindrical surface, passed through the curve perpendicular to the plane on which the projection is made with this plane.

If we know the equations of two of its projections, these equations show that the curve lies on the surfaces of two cylinders, passing through these projections, and perpendicular to their planes respectively. Hence their intersection determines the curve.

91. The term *Cylinder* is used in its most general sense, and applies to any surface generated by a right line moving parallel to itself along any curve.

92. To find the equations of a right line, passing through two given points.

Let $x'$, $y'$, $z'$, $x''$, $y''$, $z''$ : be the co-ordinates of these points. The equations of the required line will be of the form

$$x = az + \alpha$$
$$y = bz + \beta$$

$a$, $b$, $\alpha$, $\beta$, being unknown. In order that the line pass through the point whose co-ordinates are $x'$, $y'$, $z'$, it is necessary that these equations be satisfied when we substitute $x'$, $y'$ and $z'$, for $x$, $y$ and $z$. Hence

$$x' = az' + \alpha$$
$$y' = bz' + \beta.$$

For the same reason, the condition of its passing through

the point whose co-ordinates are $x''$, $y''$, $z''$, requires that we have

$$x'' = az'' + \alpha$$

$$y'' = bz'' + \beta.$$

These equations make known $a$, $b$, $\alpha$, $\beta$, and substituting their values in the equation of the straight line, it is determined. Operating upon these equations as in Art. 61, we have

$$(x - x') = a\,(z - z'), \qquad (x' - x'') = a\,(z' - z'')$$

$$(y - y') = b\,(z - z'), \qquad (y' - y'') = b\,(z' - z'')$$

from which we get

$$a = \frac{x' - x''}{z' - z''}, \qquad b = \frac{y' - y''}{z' - z''}$$

$$(x - x') = \frac{x' - x''}{z' - z''}\,(z - z'), \quad y - y' = \frac{y' - y''}{z' - z''}\,(z - z').$$

The two last equations are those of the required line, the other two make known the angles which its projection on the planes of $xz$ and $yz$ make with the axis of $z$.

93. To find the angle included between two given lines. Let

$$\left.\begin{aligned} x &= az + \alpha \\ y &= bz + \beta \end{aligned}\right\}$$ be the equations of the first lines,

$$\left.\begin{aligned} x &= a'z + \alpha' \\ y &= b'z + \beta' \end{aligned}\right\}$$ those of the second.

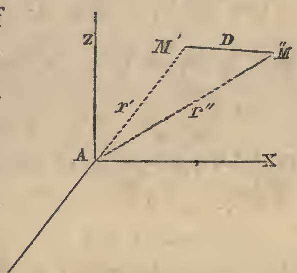
We will remark in the first place, that in space, two lines may cross each other under different angles, without meeting, and their inclination is measured in every case, by that of two lines, drawn parallel respectively to the given lines, through the same point.

Draw through the origin of co-ordinates two lines respectively parallel to those whose inclination is required their equations will be

$$\left.\begin{aligned} x &= az \\ y &= bz \end{aligned}\right\} \text{for the first,}$$

$$\left.\begin{aligned} x &= a'z \\ y &= b'z \end{aligned}\right\} \text{for the second.}$$

Take on the first any point at a distance $r'$ from the origin, the co-ordinates of this point being $x'$, $y'$, $z'$; and on the second line take another point at a distance $r''$ from the origin, and call the co-ordinates of this point $x''$, $y''$, $z''$, and let D represent the distance between these two points. In the triangle formed by the three lines $r'$, $r''$, and D, the angle V included between $r'$ and $r''$ will be (by Trigonometry), given by the formula,




$$\cos V = \frac{r'^2 + r''^2 - D^2}{2\,r'\,r''}.$$

We have only to determine $r'$, $r''$, and D.

Designating by X, Y, Z, the three angles which the first line makes with the co-ordinate axes, respectively, and by X′, Y′, Z′, those made by the second line, we have by Art. 81,

$$x' = r' \cos X, \quad y' = r' \cos Y, \quad z' = r' \cos Z$$
$$x'' = r'' \cos X', \quad y'' = r'' \cos Y', \quad z'' = r'' \cos Z'.$$

Besides, D being the distance between two points, we have

$$D^2 = (x'' - x')^2 + (y'' - y')^2 + (x'' - x')^2;$$

or

$$D^2 = x'^2 + y'^2 + z'^2 + x''^2 + y''^2 + z''^2 - 2\,(x''\,x' + y'' + y' + z''\,z').$$

Putting for $x'$, $y'$, $z'$, &c. their values in terms of the angles, we have

$$D^2 = r'^2\ \{\cos^2 X + \cos^2 Y + \cos^2 Z\} + r''^2\ \{\cos^2 X' + \cos^2 Y' + \cos^2 Z'\} - 2\,r'\,r''\,\{\cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z'\}.$$

But we have (Art. 81),

$$\cos^2 X + \cos^2 Y + \cos^2 Z = 1,\ \cos^2 X' + \cos^2 Y' + \cos^2 Z' = 1;$$

hence

$$D^2 = r'^2 + r''^2 - 2r'\,r''\,(\cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z').$$

Substituting this value of $D^2$ in the formula for the cosine V, and dividing by $r'\,r''$, we have

$$\cos V = \cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z';$$

which is the expression for the cosine of the angle formed in space.

94. We may also express cos V in functions of the coefficients $a$, $b$, $a'$, $b'$, which enter into the equations of the lines

$$x = az, \qquad x = a'z,$$

$$y = bz, \qquad y = b'z.$$

For this purpose let us consider the point which we have taken, or the first line, whose co-ordinates are $x'$, $y'$, $z'$. These co-ordinates must have between them the relations expressed by the equations of the line; hence

$$x' = az' \quad 1$$
$$y' = bz'; \quad 2$$

and as we have always for the distance $r'$ Substitute for $x'^2$ & $y'^2$ their values drawn from above equations

$$r'^2 = x'^2 + y'^2 + z'^2,$$

we have $r'^2 = a^2z'^2 + b^2z'^2 + z'^2 = z'^2(1+a^2+b^2)$, $z'^2 = \frac{r'^2}{1+a^2+b^2}$

these three equations give

$z' = \frac{r'}{\sqrt{1+a^2+b^2}}$. Substitute this value of $z'$ in 1 & 2 & we have the value of $y'$ & $x'$.

$$x' = \frac{ar'}{\sqrt{1 + a^2 + b^2}}, \quad y' = \frac{br'}{\sqrt{1 + a^2 + b^2}}, \quad z' = \frac{r'}{\sqrt{1+a^2+b^2}}$$

But we have

$$\cos X = \frac{x'}{r'}, \quad \cos Y = \frac{y'}{r'}, \quad \cos Z = \frac{z'}{r'};$$

$x = az'$
$y' = bz'$

hence

$$\cos X = \frac{a}{\sqrt{1 + a^2 + b^2}}, \quad \cos Y = \frac{b}{\sqrt{1+a^2+b^2}},$$

$$\cos Z = \frac{1}{\sqrt{1+a^2+b^2}}.$$

Reasoning in the same manner on the equations of the second line, we shall have

$$\cos X' = \frac{a'}{\sqrt{1+a'^2+b'^2}}, \quad \cos Y' = \frac{b'}{\sqrt{1+a'^2+b'^2}},$$

$$\cos Z' = \frac{1}{\sqrt{1+a'^2+b'^2}};$$

and these values being substituted in the general value of $\cos V$, it becomes

$$\cos V = \frac{1 + aa' + bb'}{\sqrt{1+a^2+b^2}\ \sqrt{1+a'^2+b'^2}}.$$

This value of $\cos V$ is double, on account of the double

sign of the radicals in the denominator. One value belongs to the acute angle, the other to the obtuse angle, which the lines we are considering make with each other.

95. The different suppositions which we make on the angle V being introduced into the general expression of cos V, we shall obtain the corresponding analytical conditions. Let V = 90°.

Cos V = $o$, and then the equation which gives the value of cos V will give

$$1 + aa' + bb' = o,$$

which is *the condition necessary that the lines be perpendicular to each other.*

96. If the lines be parallel to each other, cos V = ± 1, and this gives

$$\pm 1 = \frac{1 + aa' + bb'}{\sqrt{1 + a^2 + b^2}\ \sqrt{1 + a'^2 + b'^2}}.$$

Making the denominator disappear, and squaring both members, we may put the result under the form

$$(a' - a)^2 + (b' - b)^2 + (ab' - a'b)^2 = o.$$

But the sum of the three squares cannot be equal to zero, unless each is separately equal to zero, which gives

$$a = a', \qquad b = b', \qquad ab' = a'b.$$

The two first indicate that the projections of the lines on the planes of $xz$ and $yz$ are parallel to each other; the third is a consequence of the two others.

97. It is evident that the angles X, Y, Z, which a straight line makes with the co-ordinate axes, are complements of the angles which the same line makes with the co-ordinate

planes. Hence, if we designate by U, U′, U″, the angles which this line makes with the planes of $xz$, $yz$ and $xy$, we shall have

$$\sin U = \frac{a}{\sqrt{1 + a^2 + b^2}}, \qquad \sin U' = \frac{b}{\sqrt{1 + a^2 + b^2}},$$

$$\sin U'' = \frac{1}{\sqrt{1 + a^2 + b^2}}.$$

98. Let it be required to find the conditions necessary that two lines should intersect in space, and also find the co-ordinates of their point of intersection.

Let

$$x = az + \alpha, \qquad x = a'z + \alpha',$$

$$y = bz + \beta, \qquad y = b'z + \beta',$$

be the equations of the given lines. If they intersect, the co-ordinates of their point of intersection must satisfy the equations of these lines at the same time. Calling $x'$, $y'$, $z'$, the co-ordinates of this point, we have

1 + 2 = 4 z = 1

$$x' = az' + \alpha, \qquad x' = a'z' + \alpha',$$

$$y' = bz' + \beta, \qquad y' = b'z' + \beta'.$$

These four equations being more than sufficient to determine, the three quantities $x'$, $y'$, $z'$, will lead to an equation of condition between $a$, $b$, $\alpha$, $\beta$, $\alpha'$, $\beta'$, $a'$, $b'$, which determine the positions of the lines, and eliminating $x'$ and $y'$, we have

$$(a - a')\, z' + \alpha - \alpha' = o, \quad (b - b')\, z' + \beta - \beta' = o,$$

and afterwards $z'$, we get

$$(a - a')\,(\beta - \beta') - (\alpha - \alpha')\,(b - b') = o,$$

*which is the equation of condition that the two lines should intersect.* If this condition be fulfilled, we may, from any

three of the preceding equations, find the values of $x'$, $y'$, $z'$, and we get

$$z' = \frac{\alpha' - \alpha}{a - a'} \text{ or } z' = \frac{\beta' - \beta}{b - b'},\ x' = \frac{a\alpha' - a'\alpha}{a - a'},\ y' = \frac{b\beta' - b'\beta}{b - b'}.$$

These values become infinite when $a = a'$ and $b = b'$. The point of intersection is then at an infinite distance. Indeed, on this supposition the lines are parallel.

99. The method which has just been applied to the intersection of two straight lines, may also be used to determine the points of intersection of two curves when their equations are known. For these points being common to the two curves, their co-ordinates must satisfy at the same time, the equations of the curves. This consideration will generally give one more equation than there are unknown quantities. Eliminating the unknown quantities, we obtain an equation of condition which must be satisfied, in order that the two curves intersect.

100. Although the preceding method be correct, it is nevertheless deficient. It establishes the condition necessary for the intersection of the curves, but does not determine the *number* of intersections. To find this, let

$$x = \varphi(z), \qquad y = \psi(z),$$

be the equations of the projections of the first curve, and

$$x = \varphi'(z), \qquad y = \psi'(z),$$

those of the second, $\varphi$, $\varphi'$, $\psi$, $\psi'$, being any functions of $z$.

As these four equations must subsist at the same time for the points of intersections of the curves, we have

$$\varphi(z) = \varphi'(z)\ (1), \qquad \psi(z) = \psi'(z)\ (2).$$

Eliminating $z$ between these two equations, we shall have

the equation of condition of which we have just spoken. To comprehend the use of this equation, we must distinguish two cases: 1st. When, knowing the constants which enter into the equations of the two curves, it is required to determine their points of intersection; 2ndly. These constants being arbitrary, to establish between them the relations which will give a determinate number of points of intersection.

101. In the first case, the equations of condition (1) and (2) are entirely known, and the values of all their co-efficients are given.

Find their greatest common divisor, and put it equal to zero; we shall obtain an equation in $z$, which being resolved, will make known all the values of $z$ common to the two curves. Substitute these values, successively, in the equations of the two curves, and find those of $x$ and $y$.

Every real value for these variables, which is the same for the two curves, will indicate as many real points of intersection.

102. But if the constants which enter into the equations of the curves be arbitrary, we may profit by this indetermination, to establish between the equations (1) and (2), a common divisor of a degree, not exceeding the number of these constants. If there be but one arbitrary constant, we can establish a common divisor of the first degree; For, if $c$ be this constant, since it is arbitrary, we can have a common divisor of the form $(z - c)$. The degree of the common divisor being determined, substitute the values of $z$, obtained, by putting the divisor equal to zero, in the equations of the curves. The values of $x$ and $y$, which are real and common to the two curves, will determine whether the required number of the intersections really exist.

103. To represent these conditions geometrically; the

values of $z$, which satisfy equation (1), make known the ordinates of the points of intersection of the projection of the two curves on the plane of $zx$.

Equation (2) expresses an analogous condition with respect to their projections on the plane of $yz$. But these conditions do not determine whether the curves themselves intersect, unless the points in which the projections intersect, correspond, two and two, to the same point in space.

### *Of the Plane.*

104. We have seen that a line is characterized when we have an equation which expresses the relations between the co-ordinates of each of its points. It is the same with surfaces, and their character is determined when we have an equation between the co-ordinates $x$, $y$, and $z$, of the points which belong to it; for by giving values to two of these variables, the third can be deduced, which will give a point on the surface.

105. *The Equation of a Plane* is an equation which expresses the relations between the co-ordinates of every point of the plane.

106. Let us find this equation.

A plane may be generated by considering it as the *locus* of all the perpendiculars, drawn through one of the points of a given straight line. Let $x'$, $y'$, $z'$, be the co-ordinates of this point, we have

$$\left.\begin{array}{l} x - x' = a\,(z - z') \\ y - y' = b\,(z - z') \end{array}\right\} \text{for the equations of the given line.}$$

Those of another line drawn through the same point, will be

$$x - x' = a'\,(z - z')$$

$$y - y' = b' (z - z').$$

If these two lines be perpendicular, we have (Art. 95) the condition

$$1 + aa' + bb' = o,$$

$a'$ and $b'$ being constants for one perpendicular, but variables from one perpendicular to another. If we substitute for $a'$ and $b'$ their values drawn from the above equations, the resulting equation will express a relation which will correspond to all the perpendiculars, and this relation will be that which must exist between the co-ordinates of the plane which contains them. The elimination gives

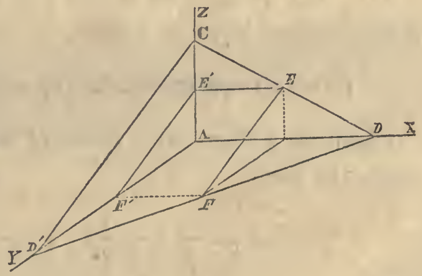
$$z - z' + a (x - x') + b (y - y') = o,$$

which is the general equation of a plane, since $a$ and $b$ are entirely arbitrary, as well as $x'$, $y'$, and $z'$.

107. If we make $x = o$, and $y = o$, we have

$$z = z' + ax' + by'$$

for the ordinate of the point C, at which the plane cuts the axis of $z$. Representing this distance by $c$, the equation of the plane becomes




$$z + ax + by - c = o,$$

and we see that it is *linear* with respect to the variables $x$, $y$, and $z$. It contains three arbitrary constants, $a$, $b$, $c$, because three conditions are, in general, necessary to determine the position of a plane in space. If $c = o$, the plane passes through the origin.

108. To find the intersection of this plane with the plane of $xz$, make $y = o$, and we have

$$y = o, \quad z + ax - c = o,$$

for the equations of the intersection CD.

The first shows that its projection on the plane of $xy$ is in the axis of $x$, and the second gives the trigonometrical tangent of the angle which it makes with the axis of $x$.

109. Making $x = o$, we obtain the intersection CD′, the equations of which are,

$$x = o, \quad x + by - c = o;$$

and $z = o$ gives

$$z = o, \quad ax + by - c = o,$$

for the equations of the intersection DD′.

The intersections CD, CD′, DD′, are called the *Traces of the Plane.*

110. The projections of the line to which this plane is perpendicular, have for their equations

$$(x - x') = a\,(z - z'), \quad (y - y') = b\,(z - z').$$

Comparing them with those of the traces CD, CD′, put under the form

$$x = -\frac{1}{a}z + \frac{c}{a}, \quad y = -\frac{1}{b} + \frac{c}{b}.$$

We see (Art. 66) that these lines are respectively perpendicular to each other. Hence, *if a plane be perpendicular to a line in space, the traces of the plane will be perpendicular to the projections of the line.*

111. Making $z = o$ in the equations of the traces CD, CD′, we have

$$z = o, \quad y = o, \quad x = \frac{c}{a},$$

and

$$z = o, \quad x = o, \quad y = \frac{c}{b},$$

for the co-ordinates of the points D, D′, in which the traces meet the axes of $x$ and $y$. These equations must satisfy the equations of the third trace DD′, because this trace passes through the points D and D′.

112. Let us put the equation of the plane under the form

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D} = o,$$

which is the same form as the preceding, if we divide by C.

We wish to show that every equation of this form is the equation of a plane.

From the nature of a plane, we know that if two points be assumed at pleasure on its surface, and connected by a straight line, this line will lie wholly in the plane. If we can prove that this property is enjoyed by the surface represented by the above equation, it will follow that this surface is a plane.

Let

$$x = az + \alpha,$$

$$y = bz + \beta,$$

be the equations of the line, and let $x'$, $y'$, $z'$, be the co-ordinates of one of the points common to the line and surface. They must satisfy the equations of the line as well as that of the surface, and we have

$$x' = az' + \alpha, \qquad y = bz' + \beta,$$

and

$$\mathrm{A}x' + \mathrm{B}y' + \mathrm{C}z' + \mathrm{D} = o.$$

Substituting for $x'$ and $y'$ their values $az + \alpha$, $bz + \beta$, we have

$$(\mathrm{A}a + \mathrm{B}b + \mathrm{C})\, z' + \mathrm{A}\alpha + \mathrm{B}\beta + \mathrm{D} = o,$$

which is the equation of condition in order that the line and surface have a common point.

Let $x''$, $y''$, $z''$, be the co-ordinates of another point common to the line and surface. We deduce the corresponding condition

$$(\mathrm{A}a + \mathrm{B}b + \mathrm{C})\, z'' + \mathrm{A}\alpha + \mathrm{B}\beta + \mathrm{D} = o.$$

Now, these two equations cannot subsist at the same time, unless we have separately

$$\mathrm{A}a + \mathrm{B}b + \mathrm{C} = o, \quad \text{and } \mathrm{A}\alpha + \mathrm{B}\beta + \mathrm{D} = o.$$

These are, therefore, the necessary conditions that the line and surface have two points common.

If the values of $a$, $b$, $\alpha$, $\beta$, are such that these two conditions are satisfied, every point of the line will be common to the surface. For, if $x'''$, $y'''$, $z'''$, be the co-ordinates of another point, in order that it be on the surface, we must have

$$(\mathrm{A}a + \mathrm{B}b + \mathrm{C})\, z''' + \mathrm{A}\alpha + \mathrm{B}\beta + \mathrm{D} = o.$$

But this equation is satisfied whenever the two others are, and consequently this point is also common to the line and surface.

As the same may be proved for every other point, it follows that every straight line which has two points in common with the surface whose equation is

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D} = o,$$

will coincide with it, and consequently this surface is a plane.

113. If we make $y = o$, we have

$$Ax + Cz + D = o$$

for the equation of the trace CD, on the plane $xz$. If the plane be perpendicular to the plane of $yz$, this trace will be parallel to the axis of $x$, and its equation will be of the form $z = a$, which requires that $A = o$, and the equation of the plane becomes

$$By + Cz + D = o.$$

We should in like manner have $B = o$, if the plane were perpendicular to the plane of $xz$. Its traces on the plane of $yz$ would be parallel to the axis of $y$, and its equation would be

$$Ax + Cz + D = o.$$

For a plane perpendicular to the plane of $xy$, we have the equation

$$Ax + By + D = o.$$

This condition requires that we have $C = o$.

We may readily see that these different forms result from the fact that $-\frac{A}{C}$, $-\frac{B}{C}$ represent the trigonometrical tangents of the angles which the traces on the planes of $xz$ and $yz$ make with the axes of $x$ and $y$.

114. There are many problems in relation to the plane which may be resolved without difficulty after what has been said. We will examine one or two of them.

115. Let it be required to find the equation of a plane passing through three given points.

Let $x'$, $y'$, $z'$; $x''$, $y''$, $z''$; $x'''$, $y'''$, $z'''$; be the co-ordinates of these points,

$$Ax + By + Cz + D = o$$

will be the form of the equation of the required plane. Since this plane must pass through the three points, we will have the relations

$$Ax' + By' + Cz' + D = o,$$

$$Ax'' + By'' + Cz'' + D = o,$$

$$Ax''' + By''' + Cz''' + D = o.$$

Then these equations will give for A, B, C, expressions of the form

$$A = A'D, \quad B = B'D, \quad C = C'D,$$

$A'$, $B'$, $C'$, being functions of the co-ordinates of the given points.

Substituting these values in the equation of the plane, we have

$$A'x + B'y + C'z + 1 = o$$

for the equation of a plane passing through three given points.

116. To find the intersection of two planes represented by the equations

$$Ax + By + Cz + D = o,$$

$$A'x + B'y + C'z + D' = o.$$

These equations must subsist at the same time for the points which are common to the two planes. We may then determine these points by combining these equations.

If we eliminate one of the variables, $z$ for example, we have

$$(AC' - A'C)\,x + (BC' - B'C)\,y + (DC' - D'C) = o.$$

This equation being of the first degree, belongs to a

straight line. It represents the equation of the projection of this intersection on the plane of $xy$.

By eliminating $x$ or $y$, we can in a similar manner find the equation of its projection on the planes of $yz$ and $xz$.

117. Generalizing this result, we may find the intersections of any surfaces whatever. For, as their equations must subsist at the same time for the points which are common, by eliminating either of the variables, the resulting equations will be those of the projections of the intersections on the co-ordinate planes.

### *Of the Transformation of Co-ordinates.*

118. We have seen that the form and position of a curve are always expressed by the analytical relations which exist between the co-ordinates of its different points. From this fact, curves have been classified into different orders from the degree of their equations.

119. Curves are called *Algebraic* or *transcendental*, according as the equations which express them are algebraic or transcendental.

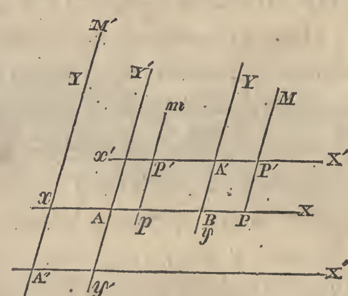
120. *Algebraic Curves* are classified from the degree of their equation, and the order of the curve is indicated by the exponent of this degree. For example, the straight line is of the *first order*, because its equation is of the first degree with respect to the variables $x$ and $y$.

121. The *discussion* of a curve consists in classifying it and determining its position and form from its equations. This discussion may be very much facilitated by means of analytical transformations, which, by simplifying the equations of the curve, enable us more readily to discover its

form and general properties. The methods used to effect this simplification consists in changing the position of the origin, and the direction of the co-ordinate axes, so that the proposed equations, when referred to them, may have the simplest form which the nature of the curve will admit of.

122. When we wish to pass from one system of co-ordinates to another, we find, for any point, the values of the old co-ordinates in terms of the new. Substituting these values in the proposed equation, it will express the relations between the co-ordinates of the same points referred to this new system. Consequently the properties of the curve will remain the same, as we have only changed the manner of expressing them.

123. The relations between the new and old co-ordinates are easily established, when the origin alone is changed without altering the direction of the axes. For, let A′ be the new origin, and A′X′, A′Y′, the new axes, parallel to the old axes, A$x$, AY. For any point M, we have



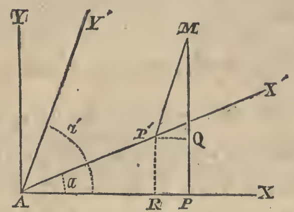

$$AP = AB + BP, \quad PM = PP' + P'M = A'B + P'M.$$

Making AB $= a$, and AB′ $= b$, and representing by $x$ and $y$ the old, and $x'$ $y'$ the new co-ordinates, these equations become

$$x = a + x', \quad y = b + y',$$

which are the equations of transformation from one system of co-ordinate axes, to another system parallel to the first.

124. To pass from one system of rectangular co-ordinates to another system oblique to the first, the origin remaining the same.




Let AY, AX be two axes at right angles to each other and AY′, A$x'$, two axes making any angle with each other. Through any point M, draw MP, MP′, respectively parallel to AY and AY′, and through P′ draw P′Q, P′R parallel to AX and A$y$, we shall have

$$x = \mathrm{AP} = \mathrm{AR} + \mathrm{P'Q}, \quad y = \mathrm{MP} = \mathrm{MQ} + \mathrm{P'R}.$$

But AR, P′R, MQ, PQ, are the sides of the right-angled triangles AP′R, P′MQ, in which AP′ $= x'$, and P′M $= y'$. We also know the angles P′AR $= \alpha$ and MP′Q $= \alpha'$. We deduce from these triangles

$$x = x' \cos\alpha + y' \cos\alpha', \qquad y = x' \sin\alpha + y' \sin\alpha',$$

which are the relations which subsist between the co-ordinates of the two systems.

125. If we wished to pass from the system whose co-ordinates are $x'$ and $y'$ to that of $x$ and $y$, we have only to deduce the values $x'$ and $y'$ from the two last equations. We find by elimination these values to be

$$x' = \frac{x \sin\alpha' - \cos\alpha'}{\sin(\alpha' - \alpha)} \qquad y' = \frac{y\cos\alpha - x\sin\alpha}{\sin(\alpha' - \alpha)}.$$

If the new axes of $x'$ and $y'$ be rectangular also, we have

$$\alpha' - \alpha = 90^\circ \text{ and } \alpha' = 90^\circ + \alpha, \quad \sin(\alpha' - \alpha) = \sin 90^\circ = 1.$$

$$\sin\alpha' = \sin(90^\circ + \alpha) = \sin\alpha\cos 90^\circ + \cos\alpha\sin 90^\circ = \cos\alpha,$$

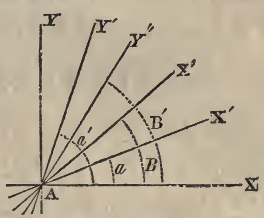
$$\cos\alpha = -\sin\alpha.$$

Substituting these values, we have for the formulas for passing from a system of rectangular co-ordinates to another system also rectangular, the origin remaining the same,

$$x = x' \cos \alpha - y' \sin \alpha, \quad y = x' \sin \alpha + y' \cos \alpha.$$

126. To pass from a system of oblique co-ordinates to another system also oblique, the origin remaining the same.

Let AX′, AY′ be the axes of $x'$, $y'$, and AX″, AY″, the new axes whose co-ordinates are $x''$, $y''$. Let us take a third system at right angles to each other as AX, AY, the co-ordinates being $x$, $y$. Calling $\alpha$, $\alpha'$, $\beta$, $\beta'$, the angles which the axes of $x'$, $y'$, $x''$, $y''$, make with the axis of $x$, we have (Art. 124) for passing from this system to the two systems of oblique co-ordinates, the formulas




$$x = x' \cos \alpha + y' \cos \alpha', \qquad y = x' \sin \alpha + y' \sin \alpha',$$
$$x = x'' \cos \beta + y'' \cos \beta', \qquad y = x'' \sin \beta + y'' \sin \beta'.$$

Eliminating $x$ and $y$ from these equations, we shall obtain the equations which will express the relations between the co-ordinates $x'$, $y'$, and $x''$, $y''$, which are

$$x' \cos \alpha + y' \cos \alpha' = x'' \cos \beta + y'' \cos \beta'$$
$$x' \sin \alpha + y' \sin \alpha' = x'' \sin \beta + y'' \sin \beta'.$$

Multiplying the first by $\sin \alpha$, and subtracting from it the second multiplied by $\cos \alpha$, we obtain the value of $y'$. Operating in the same manner, we get the value of $x'$, and the formulas become

$$x' = \frac{x'' \sin (\alpha' - \beta) + y'' \sin (\alpha' - \beta')}{\sin (\alpha' - \alpha)}$$

$$y' = \frac{x'' \sin(\beta - \alpha) + y'' \sin(\beta' - \alpha)}{\sin(\alpha' - \alpha)}$$

127. Generalizing the foregoing remarks, we may easily find the formulas for the transformation of co-ordinates in space. We have only to find the value of the old co-ordinates in terms of the new, and reciprocally. If the transformation be to a parallel system, and $a$, $b$, $c$ represent the co-ordinates of the new origin, we have the formulas

$$x = a + x', \; y = b + y', \; z = c + z',$$

in which $x$, $y$, and $z$ are the old, and $x'$, $y'$ and $z'$ the new co-ordinates.

128. Let us now suppose that the direction of the new axes is changed. As the introduction of the three dimensions of space necessarily complicates the constructions of the problems, if we can ascertain the form of the relations which must exist between the old and new co-ordinates, this difficulty may be obviated.

Now it can be proved, in general, that in passing from any system of co-ordinates, the old co-ordinates must always be expressed in linear functions of the new, and reciprocally. This has been verified in the system of co-ordinates for a plane, since the relations which we have obtained are of the first degree. To show that this must also be the case with transformations in space, let us conceive the values of $x$, $y$, $z$, expressed in any functions of $x'$, $y'$, $z'$, which we will designate by $\varphi$, $\varpi$, $\psi$, so that we have

$$x = \varphi(x', y', z'), \; y = \varpi(x', y', z'), \; z = \psi(x', y', z').$$

If we substitute these values in the equation of the plane, which is always of the form

$$Ax + By + Cz + D = o,$$

it becomes

$$A.\ \varphi\,(x', y', z',) + B.\ \varpi\,(x', y', z',) + C.\ \psi\,(x', y', z',) + D = o.$$

But the equation of the plane is always of the first degree, whatever be the direction of the rectilinear axes, to which it is referred, since the equations of its linear generatrices are always of the first degree. Hence, the preceding equations must reduce to the form

$$A'x' + B'y' + C'z' + D' = o,$$

in which $A'$, $B'$, $C'$, $D'$, are independent of $x'$, $y'$, $z'$, but dependent upon the primitive constants A, B, C, D, and the angles and distances which determine the relative positions of the two systems.

This reduction must take place whatever be the values of the primitive co-efficients A, B, C, D, and without there resulting any condition from them. Hence this reduction must exist in the functions $\varphi$, $\varpi$, $\psi$, themselves, for if it were otherwise, the terms of $\varphi$ which are multiplied by A, would not, in general, cause those of $\varpi$ and $\psi$ to disappear, which are multiplied by B and C. It would follow from this, that the powers of $x'$, $y'$, $z'$, higher than the first, would necessarily remain in the transformed equation, if they existed in the functions $\varphi$, $\varpi$, $\psi$, These functions are therefore limited by the condition that the new co-ordinates $x'$, $y'$, $z'$, exist only of the first power, and consequently the most general form which we can suppose, will be

$$x = a + mx' + m'y' + m''z',$$

$$y = b + nx' + n'y' + n''z',$$

$$z = c + px' + p'y' + p''z',$$

in which the co-efficients of $x'$, $y'$, $z'$, are unknown constants which it is required to determine. But since they are constants, their values will remain always the same, whatever be those of $x'$, $y'$, $z'$. We can then give particular values to these variables, and thus determine those of the constants. If we make

$$x' = o, \quad y' = o, \quad z' = o,$$

we have

$$x = a, \quad y = b, \quad z = c,$$

which are the co-ordinates of the new origin with respect to the old. We will suppose for more simplicity that the direction of the axes is changed, without removing the origin; the preceding formulas become under this supposition

$$x = mx' + m'y' + m''z',$$
$$y = nx' + n'y' + n''z',$$
$$z = px' + p'y' + p''z'.$$

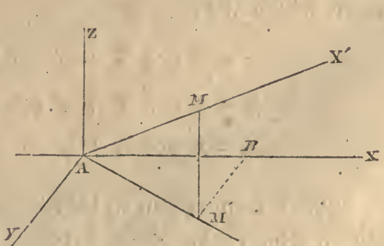
To determine the constants, let us consider the points placed on the axis of $x'$, the equations of this axis are

$$y' = o, \quad z' = o.$$

We have then for points situated on it,

$$x = mx', \quad y = nx'; \quad z = px'.$$

Let AX′ be this axis, and let the old axes AX, AY, AZ, be taken at right angles, for any point M we have AM $= x'$, MM′ $= z$, and the triangle AMM′ will give



$$z = x' \cos \text{AMM}'.$$

The angle AMM′ is that which the new axis of $x'$ makes with the old axis of $z$. Let us call it $z$, and represent by XY the angles formed by this same axis AX′, with AX and AY. We shall have for points on this axis

$$x = x' \cos \text{X}, \quad y = x' \cos \text{Y}, \quad z = x' \cos \text{Z}.$$

This result determines $n$, $m$, $p$, and gives

$$m = \cos \text{X}, \quad n = \cos \text{Y}, \quad p = \cos \text{Z}.$$

If we consider points on the axis of $y'$, whose equations are

$$x' = o, \qquad z' = o,$$

we shall have relatively to these points

$$x = m'y', \quad y = n'y', \quad z = p'y'.$$

Designating by X′, Y′, Z′, the angles which this axis forms with the axes of $x$, $y$, $z$, we have

$$m' = \cos \text{X}', \quad n' = \cos \text{Y}', \quad p' = \cos \text{Z}'.$$

Reasoning in the same manner with the axis $z'$, we have

$$m'' = \cos \text{X}'', \quad n'' = \cos \text{Y}'', \quad p'' = \cos \text{Z}'';$$

from which we get

$$\begin{aligned} x &= x' \cos \text{X} + y' \cos \text{X}' + z' \cos \text{X}'', \\ y &= x' \cos \text{Y} + y' \cos \text{Y}' + z' \cos \text{Y}'', \\ \text{Z} &= x' \cos \text{Z} + y' \cos \text{Z}' + z' \cos \text{Z}''. \end{aligned} \qquad (1).$$

**129.** We must join to these values, the equations of condition which take place between the three angles, which a

straight line makes with the three axes, and which are (Art. 81),

$$\cos^2 X + \cos^2 Y + \cos^2 Z = 1,$$
$$\cos^2 X' + \cos^2 Y' + \cos^2 Z' = 1,$$
$$\cos^2 X'' + \cos^2 Y'' + \cos^2 Z'' = 1. \qquad (2).$$

These formulas are sufficient for the transformation of co-ordinates, whatever be the angles which the new axes make with each other.

130. Should it be required that the new axes make particular angles with each other, there will result new conditions between X, Y, Z, X′ &c. which must be joined to the preceding equations. If we represent by V the angle formed by the axis of $x'$ with that of $y'$, by U that made by $y'$ with $z'$, and by W that made by $z'$ with $x'$, we have by Art. 93,

$$\cos V = \cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z',$$
$$\cos U = \cos X' \cos X'' + \cos Y' \cos Y'' + \cos Z' \cos Z'',$$
$$\cos W = \cos X \cos X'' + \cos Y \cos Y'' + \cos Z \cos Z''. \quad (3).$$

And these equations added to those of (1) and (2), will enable us in every case to establish the conditions relative to the new axes, in supposing the old rectangular.

131. If, for example, we wish the new system to be also rectangular, we shall have

$$\cos V = o, \qquad \cos U = o, \qquad \cos W = o,$$

and the second members of equations (3) will reduce to zero; then adding together the squares of $x$, $y$, $z$, we find

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2.$$

This condition must in fact be fulfilled, for in both systems the sum of the squares of the co-ordinates represents the distance of the point we are considering, from the common origin.

132. If we wished to change the direction of two of the axes only, as, for example, those of $x$ and $y$, let us suppose that they make an angle V with each other, and continue perpendicular to the axis of $z$. We have from these conditions,

$$\cos U = o, \qquad \cos W = o,$$
$$\cos X'' = o, \qquad \cos Y'' = o, \qquad \cos Z'' = 1.$$

Substituting these values in equations (3), we have

$$\cos Z' = o, \qquad \cos Z = o,$$

that is, the axes of $x'$ and $y'$ are in the plane of $xy$.

From this and equations (2), there results

$$\cos Y = \sin X, \qquad \cos Y' = X',$$

and the values of $x$, and $y$, become

$$x = x' \cos X + y' \cos X', \quad y = x' \sin X + y' \sin X';$$

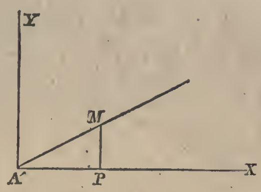
which are the same formulas as those obtained (Art. 124.)

### *Polar Co-ordinates.*

133. Right lines are not the only co-ordinates which may be used to define the position of points in space. We may employ any system of lines, either straight or curved, whose construction will determine these points.

134. For example, we may take for the co-ordinates of

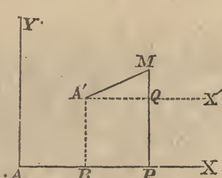
points situated in a plane, the distance AM, from a fixed point A taken in a plane, and the angle MAX, made by the line AM with any line AX drawn in the same plane. For, if we have




the angle MAP, the direction of line AM is known; and if the distance AM be also known, the position of the point M is determined.

135. The method of determining points by means of a variable angle and distance, is called a *System of Polar Co-ordinates*. The distance AM is called the *Radius Vector*, and the fixed point A the *Pole*.

136. When we know the equation of a line, referred to rectilinear co-ordinates, we may transpose it into polar co-ordinates, by determining the values of the old co-ordinates in terms of the new, and substituting them in the proposed equation. For example, let A′ be taken as the pole, whose co-ordinates are $x = a$, $y = b$. Draw A′X′ parallel to the axis of $x$, and designate the angle MA′X′ by $v$; the radius vector A′M by $r$, we have




$$AP = AB + A'Q, \quad PM = A'B + MQ,$$

or,

$$x = a + A'Q, \qquad y = b + MQ.$$

But in the right-angled triangle A′MQ, we have

$$A'Q = r \cos v, \text{ and } MQ = r \sin v.$$

Substituting these values, we have

$$x = a + r \cos v, \quad y = r \sin v, \quad (1)$$

which are the formulas for passing from rectangular co-ordinates to polar co-ordinates.

137. If the pole coincide with the origin, $a = o$, $b = o$, and we have

$$x = r \cos v, \quad y = r \sin v.$$

If the line AX′ make an angle $\alpha$ with the axis of $x$, formulas (1) will become

$$x = a + r \cos (v + \alpha), \quad y = b + r \sin (v + \alpha).$$

138. By giving to the angle $v$ every value from $o$ to $360°$, and varying the radius vector from zero to infinity, we may determine the position of every point in a plane. But from the equation

$$x = r \cos v$$

we get

$$r = \frac{x}{\cos v}.$$

Now, since the algebraic signs of the abscissa and cosine vary together, that is, are both positive in the first and fourth quadrants, and negative in the second and third, it follows that the *radius vector can never be negative*, and we conclude that should a problem lead to negative values for the radius vector, it is impossible.

139. Polar co-ordinates may also be used to determine the position of points in space. For this purpose we make use of the angle which the radius vector AM makes with its projection on the plane of $xy$, for example, and that which this projection makes with the axis of $x$. MAM′ is the first of these angles; MAP the second. Calling them $\varphi$ and $\theta$, and representing the radius vector AM by $r$, and its projection AM′ by $r'$, we have

$$x = r' \cos \varphi, \quad y = r' \sin \theta, \quad z = r \sin \theta.$$

Besides, we have

$$r' = r \cos \theta;$$

from which we get

$$x = r \cos \varphi \cos \theta, \quad y = r \cos \theta \sin \varphi, \quad z = r \sin \theta.$$

Formulas which may be applied to every point in space by attributing to the variables $\theta$, $\varphi$, and $r$, every possible value.

# CHAPTER IV.

## OF THE CONIC SECTIONS.

140. If a right cone with a circular base, be intersected by planes having different positions with respect to its axis, the curves of intersection are called *Conic Sections*. As this common mode of generation establishes remarkable analogies between these curves, we shall employ it to find their general equation.

141. Let O be the origin of a system of rectangular co-ordinates OX, OY, OZ. If the line AC at the distance OC = C from the origin, revolve about the axis OZ, making a constant angle $v$ with the plane of $xy$, it will generate the surface of a right cone with a circular base, of which C will be the vertex and CO the axis. The part CA will generate the *lower nappe*, CA′ the *upper nappe* of the cone. To find the equation of this surface.


The equation of a line passing through the point C, whose co-ordinates are

$$x = o, \quad y = o, \quad z = c,$$

is of the form (Art. 92),

$$x = a\,(z - c), \quad y = b\,(z - c);$$

the co-efficients $a$ and $b$ being constants for the same position of the generatrix, but variables from one position to another. But we have (Art. 97),

$$\sin^2 v = \frac{1}{1 + a^2 + b^2},$$

from which we obtain

$$(a^2 + b^2)\ \text{tang}^2 v = 1.$$

Substituting for $a$ and $b$, their values drawn from the equation of the generatrix. we shall have

$$(y^2 + x^2)\ \text{tang}^2 v = (z - c)^2.$$

This equation being independent of $a$ and $b$, it corresponds to every position of the line AC in the generation, it is therefore the equation of the conic surface.

142. Let this surface be intersected by a plane BOY, drawn through the origin O, and perpendicular to the plane of $xz$. Designating by $u$ the angle BOX which it makes with the plane of $xy$, its equation will be the same as that of its trace BO (Art. 113), that is

$$Z = X\ \text{tang}\ u.$$

If we combine this equation with that of the conic surface, we shall obtain the equations of the projections of the curve of intersection on the co-ordinate planes. But as the properties of the curve may be better discovered, by referring it to axes, taken in its own plane, let us find its equation referred to the two axes OB, OY, which are situated in its plane, and at right angles to each other, calling $x'$ $y'$ the

co-ordinates of any point, the old co-ordinates of which were $x$, $y$, $z$, we shall have in the right-angled triangle OPP′,

$$X = OP = x' \cos u, \quad Z = PP' = x' \sin u;$$

and since the axes of $y$ and $y'$ coincide, we shall also have

$$y = y'.$$

Substituting these values for $x$, $y$, $z$, in the equation of the surface of the cone, we shall obtain for the *equation of intersection*

$$y'^2 \operatorname{tang}^2 v + x'^2 \cos^2 u \,(\operatorname{tang}^2 v - \operatorname{tang}^2 u) + 2cx' \sin u = c^2;$$

or suppressing the accents,

$$y^2 \operatorname{tang}^2 v + x^2 \cos^2 u \,(\operatorname{tang}^2 v - \operatorname{tang}^2 u) + 2cx \sin u = c^2.$$

**143.** In order to obtain the different forms of the curves of intersection of the plane and cone, it is evident that all the varieties will be obtained by varying the angle $u$ from $o$ to 90°. Commencing then by making

$$u = o,$$

which causes the cutting plane to coincide with the plane of $xy$, the equation of the intersection becomes

$$y^2 + x^2 = \frac{c^2}{\operatorname{tang}^2 v},$$

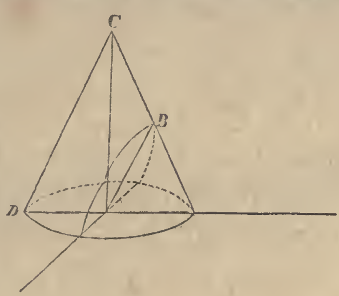
which shows that all of its points are equally distant from the axis of the cone. The intersection therefore is a circle, described about O as a centre and with a radius equal to $\dfrac{c}{\operatorname{tang} v}$.

**144.** Let $u$ increase, the plane will intersect the cone in

a re-entrant curve, so long as $u < v$, which will be found entirely on one nappe of the cone. But $u < v$ makes tang $u <$ tang $v$, and the co-efficients of $x^2$ and $y^2$ will be *positive* in the equation of intersection. This condition characterizes a class of curves, called *Ellipses*.

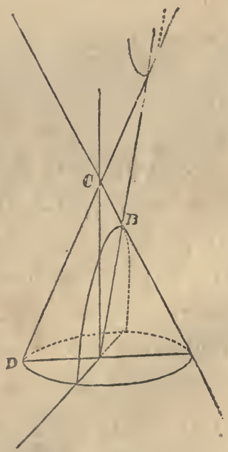
145. When $u = v$, the cutting plane is parallel to CD. The curve of intersection is found limited to one nappe of the cone, but extends indefinitely from B on this nappe. The condition $u = v$ causes the co-efficient of $x^2$ to disappear, and the general equation of intersection reduces to

$$y \text{ tang }^2 v + 2cx \sin u = c^2.$$




These curves are called *Parabolas.*

146. Finally, when $u > v$, the cutting plane intersects both nappes of the cone, and the curve of intersection will be composed of two branches, extending indefinitely on each nappe. In this case tang $u >$ tang $v$, and the co-efficient of $x^2$ becomes *negative*. This condition characterizes a class of curves called *Hyperbolas*.




147. If we suppose the cutting plane to pass through the vertex of the cone, the circle and ellipse will reduce to a

*point*, the parabola to a straight line, and the hyperbola to two straight lines intersecting at C. This becomes evident from the equations of these different curves, by making $c = o$, and also introducing the condition of $u$ being less than, equal to, or greater than, $v$.

We will now discuss each of these classes of curves, and deduce from their general equation the form and character of each variety.

### *Of the Circle.*

148. If a right cone with a circular base be intersected by a plane at a distance $c$, from the vertex, and perpendicular to the axis, we have found for the equation of intersection (Art. 143),

$$y^2 + x^2 = \frac{c^2}{\text{tang}^2 v}.$$

Representing the second member $\frac{c^2}{\text{tang}^2 v}$ by $R^2$, we have

$$x^2 + y^2 = R^2.$$

In this equation, the co-ordinates $x$ and $y$ are rectangular the quantity $\sqrt{x^2 + y^2}$ expresses therefore the distance of any point of the curve from the origin of co-ordinates (Art. 47.) The above equation shows that this distance is constant. The curve which it represents is evidently the circumference of a circle, whose centre is at the origin of co-ordinates, and whose radius is R.

149. To find the points in which the curve cuts the axis of $x$, make $y = o$, and we have

$$x = \pm R,$$

which shows that it cuts this axis in two different points, one on each side of the origin, and at a distance R from the axis of $y$. Making $x = o$, we find the points in which it cuts the axis of $y$. We get

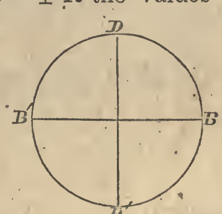
$$y = \pm \text{R},$$

which shows that the curve cuts this axis in two points, one above and the other below the axis of $x$, and at the same distance R from it.

150. To follow the course of the curve in the intermediate points, find the value of $y$ from its equation, we get

$$y = \pm \sqrt{\text{R}^2 - x^2}.$$

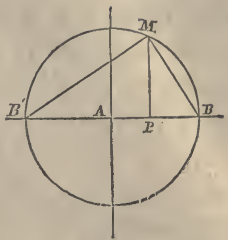
These values being equal and with contrary signs, it follows that the curve is symmetrical with respect to the axis of $x$. If we suppose $x$ positive or negative, the values of $y$ will increase as those of $x$ diminish, and when $x = o$ we have $y = \pm$ R, which gives the points D and D$'$. As $x$ increases, $y$ will diminish, and when $x = \pm$ R the values of $y$ become zero. This gives the points B and B$'$. If $x$ be taken greater than R, $y$ becomes imaginary. The curve therefore does not extend beyond the value of $x = \pm$ R.




151. The equation of the circle may be put under the form,

$$y^2 = (\text{R} + x)\,(\text{R} - x).$$

R $+ x$, and R $- x$, are the segments into which the ordinate $y$ divides the diameter. *This ordinate is therefore a mean proportional between these two segments.*




152. The equation of a line passing through the point B, whose co-ordinates are $y = o$, $x = + \text{R}$, is

$$y = a\,(x - \text{R});$$

and for a line passing through the point B′, for which $y = o$ and $x = - \text{R}$,

$$y = a\,(x + \text{R}).$$

In order that these lines should intersect on the circumference of the circle, these equations must subsist at the same time with the equation of the circle. Combining the equations with that of the circle, by multiplying the two first together, and dividing by the equation of the circle, we have first

$$y^2 = aa'\,(x^2 - \text{R}^2)\,;$$

and the division by $y^2 = (\text{R}^2 - x^2)$, gives

$$aa' = -1, \text{ or } aa' + 1 = o\,;$$

but this last equation expresses the condition that two lines should be perpendicular to each other (Art. 66) ; *hence, if two lines be drawn from the extremities of a diameter of the circle to any point of its circumference, they will be perpendicular to each other.*

153. The equation of the circle may be put under another form, by referring it to a system of a co-ordinate, whose origin is at the extremity B′ of its diameter B′B. For any point M, we have

$$\text{AP} = x = \text{B}'\text{P} - \text{B}'\text{A} = x' - \text{R}.$$

Substituting this value of $x$ in the equation $y^2 + x^2 = R^2$, we get

$$y^2 + x'^2 - 2\,Rx' = o.$$

In this equation $x' = o$ gives $y' = o$, since the origin of co-ordinates is a point of the curve. Discussing this equation as we have done the preceding, we shall arrive at the same results as those which have just been determined.

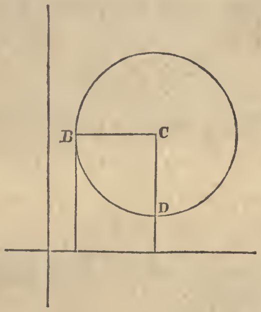
154. If the circle be referred to a system of rectangular co-ordinates taken without the circle, calling $x'$ and $y'$ the co-ordinates of the centre, and $x$ and $y$ those of any one of its points, we shall have

$$x' - x = BC, \quad y' - y = CD;$$

and calling the radius R, we have (Art. 47),

$$(x' - x)^2 + (y' - y)^2 = R^2,$$

which is the most general equation of the circle, referred to rectangular axes.




155. *To find the equation of a tangent line to the circle,* let us resume the equation

$$x^2 + y^2 = R^2.$$

Let $x''$, $y''$, be the co-ordinates of the point of tangency, they must satisfy the equation of the circle, and we have

$$x''^2 + y''^2 = R^2.$$

The equation of the tangent line will be of the form (Art. 61),

$$y - y'' = a\,(x - x'');$$

it is required to determine $a$.

For this purpose, let the tangent be regarded as a secant, and let us determine the co-ordinates of the points of intersection. These co-ordinates must satisfy the three preceding equations, since the points to which they belong are common to the line and circle. Combining these equations, by subtracting the second from the first, we have

$$y^2 - y''^2 + x^2 - x''^2 = o,$$

$$\text{or } (y - y'')\,(y + y'') + (x - x'')\,(x + x'') = o.$$

Putting for $y$, its value $y'' + a\,(x - x'')$ drawn from the equation of the line, we get

$$\{2ay'' + a^2\,(x - x'') + x + x''\}\,(x - x'') = o.$$

This equation will give the two values of $x$ corresponding to the two points of intersection. The co-ordinates of one point are obtained by putting

$$x - x'' = o,$$

which gives

$$x = x'', \text{ and } y = y'';$$

and those of the second point are made known by the equation

$$2ay'' + a^2\,(x - x'') + x + x'' = o,$$

when $a$ is given.

If now we suppose the points of intersection to approach each other, the secant line will become a tangent, when those points coincide; but this supposition makes

$$x = x'', \text{ and } y = y'';$$

and the last equation becomes

$$2ay'' + 2x'' = o,$$

from which we get

$$a = -\frac{x''}{y''}.$$

Substituting this value of $a$ in the equation of the tangent, it becomes after reduction

$$yy'' + xx'' = R^2.$$

156. The value which we have just found for $a$ being single, it follows *that but one tangent can be drawn to the circle, at a given point of the curve.*

157. A line drawn through the point of tangency perpendicular to the tangent is called a *Normal.* Its equation will be of the form

$$y - y'' = a' (x - x'').$$

The condition of its being perpendicular to the tangent gives

$$a'a + 1 = o, \text{ or } a' = -\frac{1}{a}.$$

But we have found (Art. 155),

$$a = -\frac{x''}{y''};$$

hence,

$$a' = \frac{y''}{x''}.$$

Substituting this value in the equation of the normal, it becomes

$$y - y'' = \frac{y''}{x''}(x - x'');$$

and reducing, we have

$$yx'' - y''x = o$$

for the equation of the normal line to the circle.

158. The normal line to the circle passes through its centre, which, in this case, is the origin of co-ordinates. For, if we make one of the variables equal to zero, the other will be zero also. *Hence the tangent to a circle is perpendicular to the radius drawn through the point of tangency.*

159. To draw a tangent to the circle, through a point without the circle, let $x'$ $y'$ be the co-ordinates of this point. Since it must be on the tangent, it must satisfy the equation of this line, and we have

$$y'\,y'' + x'\,x'' = \mathrm{R}^2.$$

We have besides,

$$y''^2 + x''^2 = \mathrm{R}^2.$$

These two equations will determine $x''$ and $y''$, the co-ordinates of the point of tangency, in terms of R and the co-ordinates $x'$ $y'$ of the given point. Substituting these values in the equation of the tangent, it will be determined.

The preceding equations being of the second degree, will give two values for $x''$ and $y''$. There will result consequently two points of tangency, and hence two tangents may be drawn to a circle from a given point without the circle.

160. We have seen that the equation of the circle referred to rectangular co-ordinates, having their origin at the

centre, only contains the squares of the variables $x$ and $y$, and is of the form

$$y^2 + x^2 = R^2.$$

Let us seek if there be any other systems of axes, to which, if the curve be referred, its equation will retain the same form.

Let us refer the equation of the circle to systems having the same origin, and whose co-ordinates are represented by $x'$ and $y'$. Let $\alpha$, $\alpha'$, be the angles which these new axes make with the axis of $x$. We have for the formulas of transformation (Art. 124),

$$x = x' \cos \alpha + y' \cos \alpha', \quad y = x' \sin \alpha + y' \sin \alpha'.$$

Substituting these values for $x$ and $y$ in the equation of the circle, it becomes

$$y'^2 (\cos^2\alpha' + \sin^2\alpha') + 2x'y' \cos (\alpha' - \alpha) + x'^2 (\cos^2\alpha + \sin^2\alpha) = R^2;$$

or, reducing,

$$y'^2 + 2x'y' \cos (\alpha' - \alpha) + x'^2 = R^2.$$

The form of this equation differs from that of the given equation, since it contains a term in $x'y'$. In order that this term disappear, it is necessary that the angles $\alpha$ $\alpha'$ be such that we have

$$\cos (\alpha' - \alpha) = o,$$

which gives

$$\alpha' = \alpha + 90^\circ, \text{ or } \alpha' = \alpha + 270^\circ,$$

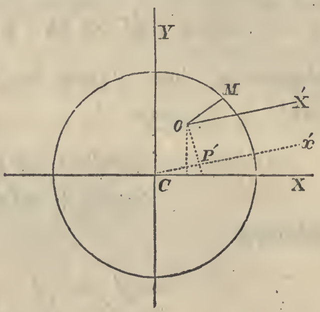
which shows that the new axes must be perpendicular to each other.

161. *Conjugate Diameters* are those diameters to which, if the equation of the curve be referred, it will contain only

the square powers of the variables. In the circle, we see that these diameters are always at right angles to each other; and as an infinite number of diameters may be drawn in the circle perpendicular to each other, it follows that there will be an infinite number of conjugate diameters.

### *Of the Polar Equation of the Circle.*

162. To find the equation of the circle referred to polar co-ordinates, let O be taken as the pole, the co-ordinates of which referred to rectangular axes are $a$ and $b$; draw OX′ making any angle $\alpha$ with the axis of $x$. OM will be the radius vector, and MOX′ the variable angle $v$. The formulas for transformation are (Art. 137),




$$x = a + r \cos (v + \alpha), \quad y = b + r \sin (v + \alpha).$$

These values being substituted in the equation of the circle

$$y^2 + x^2 = \mathrm{R}^2,$$

it becomes

$$r^2 + 2\{a \cos (v + \alpha) + b \sin (v + \alpha)\} r + a^2 + b^2 - \mathrm{R}^2 = o,$$

which is the most general polar equation of the circle.

This equation being of the second degree with respect to

$r$, will generally give two values to the radius vector. The positive values alone must be considered, as the negative values indicate points which do not exist.

163. By varying the position of the pole and the angle $v$, this equation will define the position of every point of the circle.

164. If the pole be taken on the circumference, and we call $a$, $b$, its co-ordinates, these co-ordinates must satisfy the equation of the circle, and we have the relation

$$a^2 + b^2 - R^2 = o.$$

The polar equation reduces to

$$r^2 - 2 \{a \cos (v + \alpha) + b \sin (v + \alpha) r\} = o.$$

If OX′ be parallel to the axis of $x$, the angle $\alpha$ will be zero, and this equation becomes

$$r^2 - 2 (a \cos v + b \sin v) r = o.$$

This equation may be satisfied by making $r = o$. Hence, one of the values of the radius vector is always zero, and it may be satisfied by making

$$r + 2 (a \cos v + b \sin v) = o,$$

which gives

$$r = - 2 (a \cos v + b \sin v);$$

from which we may deduce a second value for the radius vector for every value of the angle $v$.

165. If we have in this last equation $r = o$, the equation becomes

$$a \cos v + b \sin v = o,$$

$$\text{or} \quad \frac{\sin v}{\cos v} = -\frac{a}{b},$$

$$\text{or} \quad \text{tang } v = -\frac{a}{b};$$

a relation which has been before obtained (Art. 155).

166. If the pole be taken at the centre of the circle, $a$ and $b$ would be zero, and the formulas for transformation would be

$$x = r \cos v, \quad y = r \sin v.$$

## *Of the Ellipse.*

167. We have found (Art. 142,) for the general equation of intersection of the cone and plane,

$$y^2 \text{ tang}^2 v + x^2 \cos^2 u \,(\text{tang}^2 v - \text{tang}^2 u) + 2\, cx \sin u = c^2,$$

and that this equation represents a class of curves called *Ellipses*, when $u < v$. We will now examine their peculiar properties.

To facilitate the discussion, let us transfer the origin of co-ordinates to the vertex B of the curve.

For any abscissa $\text{OP}' = x$, we would have

$$x = \text{OB} - \text{BP}';$$

or calling the new abscissas $x'$,

$$x = \text{OB} - x', \text{ and } y = y'.$$

But in the triangle BOC we have the angle $\text{C} = 90° - v$,

and the angle $B = v + u$ and the side $OC = c$, and we get

$$OB = \frac{c \cos v}{\sin (v + u)},$$

from which results

$$x = \frac{c \cos v}{\sin (v + u)} - x'.$$

Substituting this value of $x$ in the equation of the curve, we have

$$y'^2 \sin^2 v + x'^2 \sin (v + u) \sin (v - u) - 2cx' \sin v$$
$$\cos v \cos u = o;$$

and suppressing the accents, we have

$$y^2 \sin^2 v + x^2 \sin (v + u) \sin (v - u) - 2cx \sin v$$
$$\cos v \cos u = o;$$

which is the general equation of the ellipse referred to the vertex B.

168. To find the points in which it meets the axis of $x$, make $y = v$, we have

$$x^2 \sin (v + u) \sin (v - u) - 2cx \sin v \cos v \cos u = o;$$

which gives for the two values of $x$,

$$x = o, \text{ and } x = \frac{2c \sin v \cos v \cos u}{\sin (v + u) \sin (v - u)},$$

which shows that it cuts the axis of $x$ in two points B and B′, one at the origin, the other at the distance

$\dfrac{2cx \sin v \cos v \cos u}{\sin (v + u) \sin (v - u)}$ on the positive side of the axis of $y$.

Making $x = o$, we have the points in which it cuts the axis of $y$. This supposition gives


$$y^2 = o.$$

This equation takes a very simple and elegant form when we introduce in it the co-ordinates of the points in which the curve cuts the axes. For, if we suppose

$$\mathrm{A}^2 = \frac{\mathrm{C}^2 \sin^2 v \cos^2 v \cos^2 u}{\sin^2 (v + u) \sin^2 (v - u)}, \text{ and}$$

$$\mathrm{B}^2 = \frac{c^2 \cos^2 v \cos^2 u}{\sin (v + u) \sin (v - u)},$$

we have only to multiply all the terms of the equation in $y$ and $x'$, by

$$\frac{c^2 \cos^2 v \cos^2 u}{\sin^2 (v + u) \sin^2 (v - u)},$$

and putting $x$ for $x'$, we have

$$y^2 \frac{c^2 \sin^2 v \cos^2 v \cos^2 u}{\sin^2 (v + u) \sin^2 (v - u)} + x^2 \frac{c^2 \cos^2 v \cos^2 u}{\sin (v + u) \sin (v - u)} =$$

$$\frac{c^2 \sin^2 v \cos^2 v \cos^2 u}{\sin (v + u) \sin (v - u)} \times \frac{c^2 \cos^2 v \cos^2 u}{\sin^2 (v + u) \sin^2 (v - u)};$$

and making the necessary substitutions, we obtain

$$\mathrm{A}^2 y^2 + \mathrm{B}^2 x^2 = \mathrm{A}^2 \mathrm{B}^2.$$

169. The quantities 2A and 2B are called the *Axes of the Ellipse.* 2A is the greater or *transverse axis*; 2B the

*conjugate* or less *axis*. The point A is the *centre* of the ellipse, and the equation

$$A^2y^2 + B^2x^2 = A^2B^2$$

is therefore the *equation of the Ellipse referred to its centre and axes.*

170. If the axes are equal we have $A = B$, and the equation reduces to

$$y^2 + x^2 = A^2,$$

which is the equation of the circle.

171. Every line drawn through the centre of the ellipse is called a *Diameter*, and since the curve is symmetrical, it is easy to see that every diameter is bisected at the centre.

172. The quantity $\frac{2B^2}{A}$ is called the parameter of the curve, and since we have

$$2A : 2B :: 2B : \frac{2B^2}{A},$$

it follows that *the parameter of the ellipse is a third proportional to the two axes.*

173. Introducing the expressions of the semi-axes A and B in the equation

$$y^2 \sin^2 v + x^2 \sin (v + u) \sin (v - u) - 2cx \sin v \cos v \cos u = o,$$

in which the origin is at the extremity of the transverse axis, by multiplying each term by the quantity

$$\frac{c^2 \cos^2 v \cos^2 u}{\sin^2 (v + u) \sin^2 (v - u)},$$

it becomes

$$A^2y^2 + B^2x^2 - 2AB^2x = o,$$

which may be put under the form

$$y^2 = \frac{B^2}{A^2} (2Ax - x^2).$$

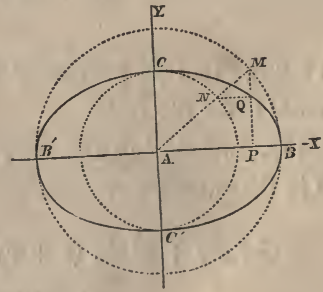
If we designate by $x'$, $y'$, $x''$, $y''$, the co-ordinates of any two points of the ellipse, we shall have

$$\frac{y'^2}{y''^2} = \frac{x' (2A - x')}{x'' (2A - x'')},$$

which shows that in the ellipse, *the squares of the ordinates are to each other as the products of the distances from the foot of each ordinate to the vertices of the curve.*

174. The equation of the ellipse referred to its centre and axes may be put under the form

$$y^2 = \frac{B^2}{A^2} (A^2 - x^2).$$




If from the point A as a centre with a radius AB = A, we describe a circumference of a circle, its equation will be

$$y^2 = A^2 - x^2.$$

Representing by $y$ and Y the ordinates of the ellipse and circle, which correspond to the same abscissa, we have

$$y = \frac{B}{A} Y.$$

According as B is less or greater than A, $y$ will be less or greater than Y, hence *if from the centre of the ellipse with radii equal to each of its axes, two circles be described, the ellipse will include the smaller and be inscribed within the large circle.*

175. From this property we deduce, 1st. That the transverse axis is the longest diameter, and the conjugate the shortest; 2ndly. When we have the ordinates of the circle described on one of the axes, to find those of the ellipse, we have only to augment or diminish the former in the ratio of B to A. This gives a method of describing the ellipse by points when the axes are known.

From the point A as a centre with radii equal to the semi-axes A and B, describe the circumferences of two circles, draw any radius ANM, and through M draw MP perpendicular to AB. The point Q will be on the ellipse, for we have

$$PQ = \frac{AN}{AM} \times PM = \frac{B}{A} \times PM,$$

or,

$$y = \frac{B}{A} \times Y,$$

as in Art. 174.

176. We have seen that for every point on the ellipse, the value of ordinate is

$$y^2 = \frac{B^2}{A^2} (A^2 - x^2).$$

For a point without the ellipse, the value of $y$ would be

greater for the same value of $x$, and for a point within, the value of $y$ would be less. Hence,

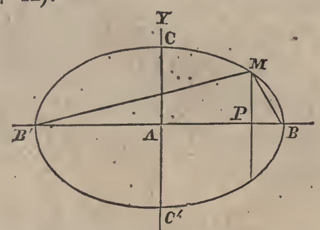
For points without the ellipse, $A^2y^2 + B^2x^2 - A^2B^2 > o$.

For points on the ellipse, $A^2y^2 + B^2x^2 - A^2B^2 = o$.

For points within the ellipse, $A^2y^2 + B^2x^2 - A^2B^2 < o$.

177. If through the point B′, whose co-ordinates are $y = o$, $x = -A$, we draw a line, its equation will be

$$y = a\,(x + A).$$




For a line passing through B, whose co-ordinates are $y = o$, $x = +A$, we have

$$y = a'\,(x - A.)$$

If it be required that these lines should intersect on the ellipse, it is necessary that these equations subsist at the same time with the equation of the ellipse. Multiplying them together, we have

$$y^2 = -aa'\,(A^2 - x^2)\,;$$

and in order that this equation agree with that of the ellipse,

$$y^2 = \frac{B^2}{A^2}\,(A^2 - x^2),$$

we must have

$$-aa' = \frac{B^2}{A^2}, \quad \text{or } aa' = -\frac{B^2}{A^2},$$

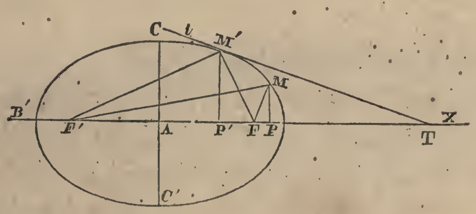
which establishes a constant relation between the angles formed by the chords drawn from the extremities of the transverse axis with this axis. In the circle B = A, and this relation becomes

$$aa' = -1,$$

as we have seen (Art. 152).

178. The lines which are drawn from the extremities of any diameter of a curve intersecting on the curve, are called *Supplementary Chords.* When the relation which has just been established (Art. 177) takes place between the angles which any two lines form with the axis of $x$, these lines are supplementary chords of an ellipse, the ratio of whose axis is $\frac{A}{B}$.

179. As we proceed in the examination of the properties of the ellipse, we are struck with the great analogy between this curve and the circle. We may trace this analogy farther. In the circle we have seen that all the points of its circumference are equally distant from the centre. Although this property does not exist in the ellipse, we find something analogous to it; for, if on the transverse axis we take two points F F′ whose abscissas are $\pm\sqrt{A^2 - B^2}$, the sum of the distances of these points to the same point of the curve is always constant and equal to the transverse axis.




To prove this, let $x$ and $y$ be the co-ordinates of any point M of the ellipse; represent the abscissas of the points F F′ by $\pm x'$. Calling D the distance MF, or MF′, we have

$$D^2 = y^2 + (x - x')^2.$$

Putting for $y$ its value drawn from the equation of the

ellipse, and substituting for $x'^2$ its value $A^2 - B^2$, this expression becomes

$$D^2 = B^2 - \frac{B^2x^2}{A^2} + x^2 - 2xx' + A^2 - B^2 =$$

$$\frac{A^2 - B^2}{A^2} x^2 - 2xx' + A^2;$$

or, substituting for $A^2 - B^2$ its value $x'^2$,

$$D^2 = \frac{x^2x'^2}{A^2} - 2xx' + A^2 = \left(A - \frac{xx'}{A}\right)^2.$$

Extracting the square root of both members, we have

$$D = \pm \left(A - \frac{xx'}{A}\right).$$

Taking the positive sign, and substituting for $x'$ its two values $\pm \sqrt{A^2 - B^2}$, we have for the distance MF, or MF′,

$$MF = A - \frac{x\sqrt{A^2 - B^2}}{A}, \quad MF' = A + \frac{x\sqrt{A^2 - B^2}}{A}.$$

Adding these values together, we get

$$MF + MF' = 2A,$$

which proves *that the sum of the distances of any point of the ellipse to the points* F F′ *is constant and equal to the transverse axis.*

180. The points F, F′, are called the *Foci* of the ellipse, and their distance $\pm \sqrt{A^2 - B^2}$ to the centre of the ellipse is called the *Eccentricity*. When $A = B$, the eccentricity $= o$. The foci in this case unite at the centre, and the ellipse becomes a circle. The maximum value of the eccentricity is when it is equal to the semi-transverse axis.

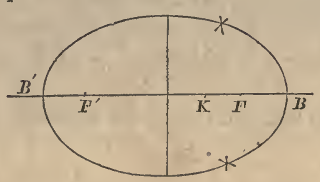
In this supposition $B = o$, and the ellipse becomes a right line.

Making $x = \pm \sqrt{A^2 - B^2}$ in the equation of the ellipse, we find

$$y = \pm \frac{B^2}{A}, \quad \text{or} \quad 2y = \pm \frac{2B^2}{A},$$

which proves that *the double ordinate passing through the focus is equal to the parameter.*

181. The property demonstrated (Art. 179) leads to a very simple construction for the ellipse. From the point B lay off any distance BK on the axis BB′. From the point F as a centre with a radius equal to BK, describe an arc of a circle; and from F′ as a centre with a radius B′K, describe another arc. The point M where these arcs intersect, is a point of the ellipse. For




$$MF + MF' = 2A.$$

When we wish to describe the ellipse mechanically, we fix the extremities of a chord whose length is equal to the transverse axis, at the foci F, F′, and stretch it by means of a pin, which as it moves around describes the ellipse.

182. To find the equation of a tangent line to the ellipse, let us resume its equation,

$$A^2y^2 + B^2x^2 = A^2 B^2.$$

Let $x''$, $y''$, be the co-ordinates of the point of tangency, they will verify the relation,

$$A^2y''^2 + B^2x''^2 = A^2 B^2.$$

The tangent line passing through this point, its equation will be of the form

$$y - y'' = a(x - x'').$$

It is required to determine $a$.

To do this, we will find the points in which this line considered as a secant meets the curve. For these points the three preceding equations must subsist at the same time. Subtracting the two first from each other, we have

$$A^2(y - y'')(y + y'') + B^2(x - x'')(x + x'') = o.$$

Putting for $y$ its value $y'' + a(x - x'')$ drawn from the equation of the line, we find

$$(x - x'')\{A^2(2ay'' + a^2(x - x'')) + B^2(x + x'')\} = o.$$

This equation may be satisfied by making

$$x - x'' = o,$$

which gives

$$x = x'',$$

from which we get

$$y = y'';$$

and also by making

$$A^2\{2ay'' + a^2(x - x'')\} + B^2(x + x'') = o.$$

Now when the secant becomes a tangent, we must have $x = x''$, which gives

$$A^2ay'' + B^2x'' = o;$$

hence

$$a = -\frac{B^2x''}{A^2x''}.$$

Substituting this value of $a$ in the equation of the tangent, it becomes

$$y - y'' = -\frac{B^2x''}{A^2y''}(x - x'');$$

or reducing, and recollecting that $A^2y''^2 + B^2x''^2 = A^2 B^2$, we have

$$A^2yy'' + B^2xx'' = A^2 B^2$$

for the *equation of the tangent line to the ellipse.*

183. If through the centre and the point of tangency we draw a diameter, its equation will be of the form

$$y' = a''x'',$$

from which we get

$$a' = \frac{y''}{x''}.$$

But we have just found the value of $a$, corresponding to the tangent line, to be

$$a = -\frac{B^2x''}{A^2y''}.$$

Multiplying these values of $a$ and $a'$ together, we find

$$aa' = -\frac{B^2}{A^2}.$$

This relation being the same as that found in Art. 177, shows that the tangent and the diameter passing through the point of tangency, have the property of being the sup-

plementary chords of an ellipse, whose axes have the same ratio $\frac{A}{B}$.

184. This furnishes a very simple method of determining the direction of the tangent. For if we draw any two supplementary chords, and designate by $\alpha$, $\alpha'$, the trigonometrical tangents of the angles which they make with the axis, we have always between them the relation

$$\alpha\,\alpha' = \frac{B^2}{A^2}.$$

We may draw one of these chords parallel to the diameter, passing through the point of tangency. In this case we have

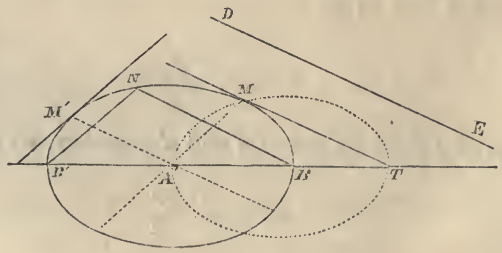
$$a' = \alpha',$$

from which results also

$$a = \alpha;$$

that is, the other chord will be parallel to the tangent.

185. To draw a tangent through a point M taken on the




ellipse, draw through this point AM, and through the extremity B′ of the axis BB′ draw the chord B′N parallel to

AM; MT parallel to BN will be the tangent required. We see, by this construction also, that if we draw the diameter AM′ parallel to the chord BN, or to the tangent MT, the tangent at the point M′ will be parallel to the chord B′N, or to the diameter AM.

186. When two diameters are so disposed that the tangent drawn at the extremity of one is parallel to the other, they are called *Conjugate Diameters*. It will be shown presently that these diameters enjoy the same property in the ellipse as those demonstrated for the circle (Art. 161).

187. To find the point in which the tangent meets the axis of $x$, make $y = o$ in its equation

$$A^2yy'' + B^2xx'' = A^2B^2,$$

and we obtain for the value of $x$

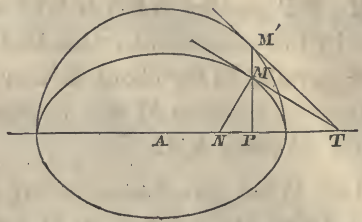
$$x = \frac{A^2}{x''},$$

which is the value of AT. If we subtract from this expression $AP = x''$, we shall have the distance PT, from the foot of the ordinate to the point in which the tangent meets the axis of $x$. This distance is called the *subtangent*. Its expression is

$$PT = \frac{A^2 - x''^2}{x''}.$$

This value being independent of the axis B, suits every ellipse whose semi-transverse axis is A, and which is concentric with the one we are considering. It therefore corresponds to the circle, described from the centre of this ellipse with a radius equal to A Hence, extending the

ordinate MP, until it meets the circle at M$'$, and drawing through this point the tangent M$'$T, MT will be tangent to the ellipse at the point M. This construction applies equally to the conjugate axis, on which the expression for the subtangent would be independent of A.




188. To find the equation of a normal to the ellipse, its equation will be of the form

$$y - y'' = a' \, (x - x'').$$

The condition of its being perpendicular to the tangent, for which we have (Art. 182),

$$a = -\frac{B^2 x''}{A^2 y''},$$

requires that there exist between $a$ and $a'$ the condition

$$aa' + 1 = o,$$

which gives

$$a' = \frac{A^2 y''}{B^2 x''}.$$

This value being substituted in the equation for the normal, gives

$$y - y'' = \frac{A^2 y''}{B^2 x''} \, (x - x'').$$

189. To find the point in which the normal meets the axis of $x$, make $y = o$ in this equation. It gives

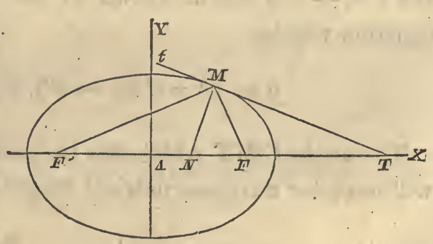
$$x = \frac{A^2 - B^2}{A^2} \cdot x''.$$

This is the value of AN. Subtracting it from AP, which is represented by $x''$, we shall have the distance from the foot of the ordinate to the foot of the normal. This distance is the *subnormal*, and its value is found to be

$$PN = \frac{B^2 x''}{A^2}.$$

190. The equation of the ellipse being symmetrical with respect to its axes, the properties which have just been demonstrated for the transverse, will be found applicable also to the conjugate axis.

191. The directions of the tangent and normal in the ellipse have a remarkable relation with those of the lines, drawn from the two foci to the point of tangency. If from the focus F, for which $y = o$ and $x = \sqrt{A^2 - B^2}$, we draw a straight line to the point of tangency, its equation will be of the form




$$y - y'' = \alpha (x - x'').$$

If we make for more simplicity $\sqrt{A^2 - B^2} = c$, the condition of passing through the focus will give

$$\alpha = -\frac{y''}{c - x''}.$$

But we have for the trigonometrical tangent which the tangent line makes with the axis of $x$ (Art. 182),

$$a = -\frac{B^2x''}{A^2y''}.$$

The angle FMT which the tangent makes with the line drawn from the focus, will have for a trigonometrical tangent (Art. 64,)

$$\frac{a - \alpha}{1 + a\alpha}.$$

Putting for $a$ and $\alpha$ their values, it reduces to

$$\frac{A^2y'^2 + B^2x''^2 - B^2cx''}{A^2cy'' - (A^2 - B^2)\,x''\,y''},$$

which reduces to

$$\frac{B^2}{cy''},$$

in observing that the point of tangency is on the ellipse, and that $A^2 - B^2 = c^2$.

In the same manner, if from the focus F′, for which $y = o$ and $x = - c$, a line be drawn to the point of tangency, its equation will be

$$y - y'' = \alpha' \,(x - x''),\ \alpha' = \frac{y''}{c - x''}.$$

The angle F′MT which this line makes with the tangent, will have for a trigonometrical tangent,

$$\frac{a - \alpha'}{1 + a\alpha'} = - \frac{B^2}{cy''},$$

when we put for $a$ and $\alpha'$ their values.

The angle FMT, F′MT, having their trigonometrical tangents equal, and with contrary signs, are supplements of each other, hence

$$\text{FMT} + \text{F}'\text{MT} = 180^\circ;$$

but

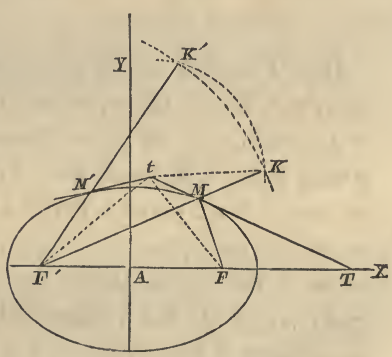
$$FMT + F'Mt' = 180°,$$

hence

$$FMT = F'MT;$$

which shows that *in the ellipse, the lines drawn from the foci to the point of tangency, make equal angles with the tangent,* and it follows from this, *that the normal bisects the angle formed by the lines drawn from the point to the same point of the curve.*

192. The property just demonstrated, furnishes a very simple construction for drawing a tangent line to the ellipse through a given point. Let M be the point at which the tangent is to be drawn. Draw FM, F′M, and produce F′M a quantity MK = FM. Joining K and F, the line MT, perpendicular to FK, will be the tangent required; for from this construction, the angles TMF, TMK, FMT, are equal to each other.




193. If the given point be without the ellipse, as at $t$, then from the point F′ as a centre, with a radius F′K = 2A describe an arc of a circle; from the point $t$ as a centre, with a radius $t$F, describe another arc, cutting the first in K. Drawing F′K, the point M will be the point of tangency, and joining M and $t$, M$t$ will be the tangent required. For, from the construction, we have $t$F = $t$K. Besides F′M + FM = 2A and F′M + MK = 2A. Hence

$$MF = MK.$$

The line $Mt$ is then perpendicular at the middle of FK. The angles FMT, $F'Mt$ are then equal, and $tMT$ is tangent to the ellipse.

194. The circles described from the points $F'$ and $t$ as centres, cutting each other in two points, two tangents may be drawn from the point $t$ to the ellipse.

### *Of the Ellipse referred to its Conjugate Diameters.*

195. There is an infinite number of systems of oblique axes, to which, if the equation of the ellipse be referred, it will contain only the square powers of the variables. Supposing in the first place, that its equation admits of this reduction, it is easy to see that the origin of the system must be at the centre of the ellipse. For, if we consider any point of the curve, whose co-ordinates are expressed by $+x', +y'$, since the transformed equation must contain only the squares of these variables, it is evident it will be satisfied by the points whose co-ordinates are $+x', -y'$; $-x', +y'$; that is, by the points which are symmetrically situated in the four angles of the co-ordinate axes. Hence every line drawn through this origin will be bisected at this point, a property which, in the ellipse, belongs only to its centre, since it is the only point around which it is symmetrically disposed.

The oblique axes here supposed will always cut the ellipse in two diameters, which will make such an angle with each other as to produce the required reduction. These lines are called *Conjugate Diameters*, which, besides

the geometrical property mentioned in Art. 186, possess the analytical property of reducing the equation of the curve to those terms which contain only the square powers of the variables.

196. The equation of the ellipse referred to its centre and axes is

$$A^2y^2 + B^2x^2 = A^2B^2.$$

To ascertain whether the ellipse has many systems of conjugate diameters, let us refer this equation to a system of oblique co-ordinates, having its origin at the centre. The formulas for transformation are (Art. 124),

$$x = x' \cos \alpha + y' \cos \alpha', \quad y = x' \sin \alpha + y' \sin \alpha'.$$

Substituting these values for $x$ and $y$ in the equation of the ellipse, it becomes

$$\left\{ \begin{array}{c} (A^2 \sin^2 \alpha' + B^2 \cos^2 \alpha')\, y'^2 + (A^2 \sin^2 \alpha + B^2 \cos^2 \alpha) \\ x'^2 + 2(A^2 \sin \alpha \sin \alpha' + B^2 \cos \alpha \cos \alpha')\, x'\, y' \end{array} \right\} = A^2\, B^2.$$

In order that this equation reduce to the same form as that when referred to its axes, it is necessary that the term containing $x'\, y'$ disappear. As $\alpha$ and $\alpha'$ are indeterminate, we may give to them such values as to reduce its coefficient to zero, which gives the condition

$$A^2 \sin \alpha \sin \alpha' + B^2 \cos \alpha \cos \alpha' = o,$$

and the equation of the ellipse becomes

$$(A^2 \sin^2\alpha' + B^2 \cos^2\alpha')\, y'^2 + (A^2 \sin^2\alpha + B^2 \cos^2\alpha) \\ x'^2 = A^2\, B^2.$$

197. The condition which exists between $\alpha$ and $\alpha'$ is not sufficient to determine both of these angles. It makes known one of them, when the other is given. We may

then assume one at pleasure, and consequently *there exists an infinite number of conjugate diameters.*

198. The axes of the ellipse enjoy the property of being conjugate diameters, for the relation between $\alpha$ and $\alpha'$ is satisfied when we suppose $\sin\alpha = o$, and $\cos\alpha' = o$, which makes the axis of $x'$ coincide with that of $x$, and $y'$ with that of $y$. These suppositions reduce the equation to the same form as that found for the ellipse referred to its axes. Or, these conditions may be satisfied by making $\sin\alpha' = o$, and $\cos\alpha = o$, which will produce the same result, only $x'$ will become $y$, and $y'$ $x$.

199. The axes are the only systems of conjugate diameters at right angles to each other. For, if we have others, they must satisfy the condition

$$\alpha' - \alpha = 90^\circ, \text{ or } \alpha' = 90^\circ + \alpha,$$

which gives

$$\sin\alpha' = \sin 90^\circ \cos\alpha + \cos 90^\circ \sin\alpha = + \cos\alpha,$$

$$\cos\alpha' = \cos 90^\circ \cos\alpha - \sin 90^\circ \sin\alpha = - \sin\alpha;$$

but these values being substituted in the equation of condition

$$A^2 \sin\alpha \sin\alpha' + B^2 \cos\alpha \cos\alpha' = o,$$

it becomes

$$(A^2 - B^2) \sin\alpha \cos\alpha = o,$$

which can only be satisfied for the ellipse by making $\sin\alpha = o$, or $\cos\alpha = o$, suppositions which reduce to the two cases just considered.

200. If we make $A^2 - B^2 = o$, we shall have $A = B$, the ellipse will become a circle, and the equation of condition being satisfied, whatever be the angle $\alpha$, it follows *that all the conjugate diameters of the circle are perpendicular to each other.*

201. Making, successively, $x' = o$, and $y' = o$, we shall have the points in which the curve cuts the diameters to which it is referred. Calling these distances A′ and B′, we find

$$A'^2 = \frac{A^2 B^2}{A^2 \sin^2\alpha + B^2 \cos^2\alpha}, \quad B'^2 = \frac{A^2 B^2}{A^2 \sin^2\alpha' + B^2 \cos^2\alpha'},$$

and the equation of the ellipse becomes

$$A'^2 y'^2 + B'^2 x'^2 = A'^2 B'^2,$$

2A′ and 2B′ representing the two conjugate diameters.

202. The parameter of a diameter is the third proportional to this diameter and its conjugate; $\frac{2B'^2}{A'}$ is therefore the parameter of the diameter 2A′, and $\frac{2A'^2}{B'}$ is that of its conjugate 2B′.

203. If we multiply the values of $A'^2$ and $B'^2$ (Art. 201) together, we get

$$A'^2 B'^2 = \frac{A^4 B^4}{A^4 \sin^2\alpha' \sin^2\alpha + A^2 B^2 (\sin^2\alpha \cos^2\alpha' + \cos^2\alpha \sin^2\alpha') + B^4 \cos^2\alpha \cos^2\alpha'},$$

which may be put under the form

$$A'^2 B'^2 = \frac{A^4 B^4}{(A^2 \sin\alpha' \sin\alpha + B^2 \cos\alpha' \cos\alpha)^2 + A^2 B^2 \sin^2(\alpha' - \alpha)}$$

But we have, from Art. 199,

$$A^2 \sin\alpha' \sin\alpha + B^2 \cos\alpha' \cos\alpha = o,$$

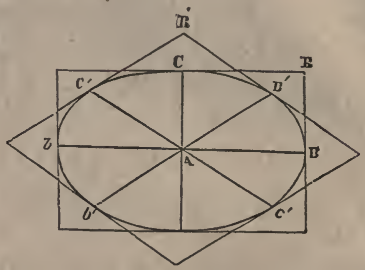
and reducing the other terms of the fraction, we have

$$A^2 B'^2 = \frac{A^2 B^2}{\sin^2(\alpha' - \alpha)},$$

which gives

$$AB = A'B' \sin(\alpha' - \alpha).$$

$(\alpha' - \alpha)$ is the expression of the angle $B'AC'$ which the two conjugate diameters make with each other. $A'B' \sin(\alpha' - \alpha)$ expresses therefore the area of the parallelogram $Ac'R'B'$. This area being equal to the rectangle $AcRB$ formed on the axes, we conclude, *that in the ellipse, the parallelogram constructed on any two conjugate diameters is equivalent to the rectangle on the axes.*




204. The equation of condition between the angles $\alpha$ and $\alpha'$ being divided by $\cos\alpha \cos\alpha'$, becomes

$$A^2 \operatorname{tang} \alpha \operatorname{tang} \alpha' + B^2 = o. \qquad (1).$$

We may easily eliminate by means of this equation the angle $\alpha'$ from the value of $B'^2$, or the angle $\alpha$ from $A'^2$. For this purpose we have only to introduce the tangents of the angles instead of their sines and cosines. Since we have always

$$\sin^2\alpha = \frac{\operatorname{tang}^2\alpha}{1 + \operatorname{tang}^2\alpha}; \quad \cos^2\alpha = \frac{1}{1 + \operatorname{tang}^2\alpha};$$

$$\sin^2\alpha' = \frac{\operatorname{tang}^2\alpha'}{1 + \operatorname{tang}^2\alpha'}; \quad \cos^2\alpha' = \frac{1}{1 + \operatorname{tang}^2\alpha'}.$$

Substituting these values for $A'^2$ and $B'^2$, we have

$$A'^2 = \frac{A^2 B^2 (1 + \operatorname{tang}^2\alpha)}{A^2 \operatorname{tang}^2\alpha + B^2}; \quad B'^2 = \frac{A^2 B^2 (1 + \operatorname{tang}^2\alpha')}{A^2 \operatorname{tang}^2\alpha' + B^2}.$$

To eliminate $\alpha'$ we have only to substitute for tang $\alpha'$ its value deduced from equation (1), and after reduction, the value of $B'^2$ becomes

$$B'^2 = \frac{A^4 \text{ tang}^2\alpha + B^4}{A^2 \text{ tang}^2\alpha + B^2}.$$

Adding this equation to the value of $A'^2$, the common numerator

$$A^2 B^2 + A^2 B^2 \text{ tang}^2\alpha + A^4 \text{ tang}^2\alpha + B^4$$

may be put under the form

$$B^2 (A^2 + B^2) + A^2 \text{ tang}^2\alpha (B^2 + A^2),$$

$$\text{or } (A^2 + B^2)(A^2 \text{ tang}^2\alpha + B^2),$$

and the same after reduction becomes

$$A'^2 + B'^2 = A^2 + B^2;$$

that is, *in the ellipse the sum of the squares of any two conjugate diameters is always equal to the sum of the squares of the two axes.*

205. The three equations

$$A^2 \text{ tang } \alpha \text{ tang } \alpha' + B^2 = o,$$

$$AB = A'B' \sin(\alpha' - \alpha),$$

$$A^2 + B^2 = A'^2 + B'^2,$$

suffice to determine three of the quantities $A, B, A', B', \alpha, \alpha'$, when the other three are known. They may consequently serve to resolve every problem relative to conjugate diameters, when we know the axes, and reciprocally.

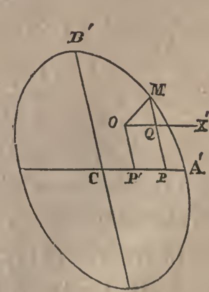
206. Comparing the first of these equations with the relations found in Art. 177; when two lines are drawn from

the extremities of the transverse axis to a point of the ellipse, we see that the angles $\alpha$, $\alpha'$, satisfy this condition. It is then always possible to draw two supplementary chords from the vertices of the transverse axis, which shall be parallel to two conjugate diameters.

207. From this results a simple method of finding two conjugate diameters, which shall make a given angle with each other, when we know the axes. On one of the axes describe a circle capable of containing the given angle. Through one of the points in which it cuts the ellipse draw supplementary chords to this axis. They will be parallel to the diameters sought, and drawing parallels through the centre of the ellipse, we shall have these diameters. If the given angle exceed the limit assigned for conjugate diameters, the problem is impossible.

### *Of the Polar Equation of the Ellipse, and of the measure of its surface.*

208. To find the polar equation of the Ellipse, let $o$ be taken as the pole, the co-ordinates of which are $a$ and $b$. The formulas for transformation are (Art. 136),




$$x = a + r \cos v, \quad y = b + r \sin v.$$

Substituting these values of $x$ and $y$, in the equation of the ellipse,

$$A^2y^2 + B^2x^2 = A^2B^2,$$

it becomes

$$\left.\begin{array}{l} A^2 \sin^2 v \\ + B^2 \cos^2 v \end{array}\right| \begin{array}{l} r^2 + 2A^2 b \sin v \\ + 2B^2 a \cos v \end{array} \left| \; r + A^2 b^2 + B^2 a^2 - A^2 B^2 = o, \right.$$

which is the polar equation of the ellipse.

209. If the pole be taken at the centre of the ellipse, we shall have

$$a = o, \text{ and } b = o\,;$$

and the equation becomes

$$(A^2 \sin^2 v + B^2 \cos^2 v)\; r = A^2 B.$$

210. If the pole be taken on the curve, this condition would require that

$$A^2 b^2 + B^2 a^2 - A^2 B^2 = o,$$

and the polar equation would reduce to

$$(A^2 \sin^2 v + B^2 \cos^2 v)\; r + (2A^2 b \sin v + 2B^2 a \cos v)\; r = o.$$

The results in this and the last article may be discussed in the same manner as in the polar equation of the circle.

211. Let us now suppose the pole to be at one of the foci, the co-ordinates of which are $b = o$, $a = + \sqrt{A^2 - B^2}$. These values being substituted in the general polar equation, it becomes

$$(A^2 \sin^2 v + B^2 \cos^2 v)\; r^2 + 2B^2 a \cos v.\, r = B^4.$$

Resolving this equation with respect to $r$, the quantity under the radical becomes

$$B^4\; (A^2 \sin^2 v + B^2 \cos^2 v) + B^4 a^2 \cos^2 v\,;$$

and putting for $a^2$ its value $A^2 - B^2$, it reduces to

$$A^2B^4 (\sin^2 v + \cos^2 v), \text{ or } A^2B^4;$$

and we have for the two values of $r$,

$$r = -\frac{B^2 (a \cos v - A)}{A^2 \sin^2 v + B^2 \cos^2 v},$$

$$\text{and } r = -\frac{B^2 (\cos v + A)}{A^2 \sin^2 v + B^2 \cos^2 v},$$

which may be put under another form, for we have

$$A^2 \sin^2 v + B^2 \cos^2 v = A^2 - (A^2 - B^2) \cos^2 v = A^2 - a^2 \cos^2 v$$

$$= (A - a \cos v)(A + a \cos v).$$

Making the substitutions, and reducing, we have

$$r = \frac{B^2}{A + a \cos v}, \qquad r = -\frac{B^2}{A - a \cos v}.$$

212. If now the pole be at the focus F, for which $a$ is positive and less than A, as the cos $v$ is less than unity, the product $a \cos v$ will be positive and less than A, so that whatever sign cos $v$ undergoes in the different quadrants, $A + a \cos v$, and $A - a \cos v$, will be both positive. The first value of $r$ will then be always positive and give real points of the curve, while the second will be always negative, and must be rejected (Art. 138). The same thing takes place at the focus F′, for although $a$ is negative in this case, $a \cos v$ will be always less than A, and the denominators of the two values will be positive. The first value alone will give real points of the curve.

213. If, for more simplicity, we make

$$\frac{A^2 - B^2}{A^2} = e^2,$$

we shall have

$$B^2 = A^2(1 - e^2), \text{ and } a = \pm Ae.$$

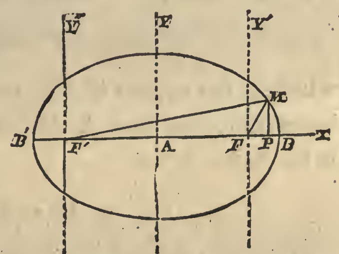
These values being substituted in the positive value of $r$, gives

$$r = \frac{A(1 - e^2)}{1 + e \cos v}, \quad r = \frac{A(1 - e^2)}{1 - e \cos v}.$$

These formulas are of frequent use in Astronomy.

214. In the preceding discussion we have deduced from the equation of the ellipse, all of its properties; reciprocally one of its properties being known we may find its equation.

For example, let it be required to find the curve, the sum of the distances of each of its points to two given points being constant and equal to 2A.




Let F, F', be the two given points, and A the middle of the line FF' the origin of co-ordinates. Represent FF' by $2c$. Suppose M to be a point of the curve, for which $AP = x$, $PM = y$, and designate the distances FM, F'M, by $r$, $r'$. We shall have

$$r^2 = y^2 + (c - x)^2; \; r'^2 = y^2 + (c + x)^2$$

$$r + r' = 2A.$$

Adding the two first equations together, and then subtracting the same equations, we shall have

$$r^2 + r'^2 = 2(y^2 + x^2 + c^2), \; r'^2 - r^2 = 4cx.$$

The second equation may be put under the form

$$(r' - r)(r' + r) = 4cx.$$

Substituting for $r' + r$ its value 2A, we get

$$r' - r = \frac{2cx}{A},$$

from which we deduce

$$r' = A + \frac{cx}{A}, \qquad r = A - \frac{cx}{A}.$$

Putting these values in the equation whose first member is $r'^2 + r^2$, we have

$$A^2 - \frac{cx^2}{A^2} = y^2 + x^2 + c^2,$$

$$\text{or } A^2 (y^2 + x^2) - c^2x^2 = A^2 (A^2 - c^2).$$

When we make $x = o$, this equation gives

$$y^2 = A^2 - c^2.$$

which is the square of the ordinate at the origin. As $c$ is necessarily less than A, this ordinate is real, and representing it by B, we have

$$B^2 = A^2 - c^2.$$

If we find the value of $c$ from this result, and substitute it in the equation of the curve, we have

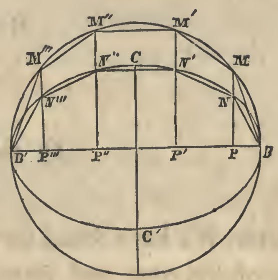
$$A^2y^2 + B^2x^2 = A^2B^2,$$

which is the equation of the ellipse referred to its centre and axes.

215. We may readily find the expression for the area of the ellipse. For we have seen (Art. 174) that if a circle be described on the transverse axis as a diameter, the relation between the ordinates of the circle and ellipse will be

$$\frac{y}{Y} = \frac{B}{A}.$$

The areas of the ellipse and circle are to each other in the same ratio of B to A.




To prove this, inscribe in the circumference BMM′B′ any polygon, and from each of its angles draw perpendiculars to the axis BB′. Joining the points in which the perpendiculars cut the ellipse, an interior polygon will be formed. Now the area of the trapezoid PNP′N is

$$\left(\frac{\text{PN} + \text{P}'\text{N}'}{2}\right) \text{PP}', \text{ or } (x - x') \frac{(y + y')}{2}.$$

The trapezoid PMP′M in the circle has for a measure

$$\frac{(\text{PM} + \text{P}'\text{M}')}{2} \text{P}\delta', \text{ or } (x - x') \frac{(y + y')}{2}.$$

These trapezoids will then be to each other in the constant ratio of B to A. The surfaces of the inscribed polygons will also be in the same ratio, and as this takes place, whatever be the number of sides of the polygons, this ratio will be that of their limits. Designating the areas of the ellipse and circle by $s$ and S, we will have

$$\frac{s}{\text{S}} = \frac{\text{B}}{\text{A}};$$

that is, the area of the ellipse is to that of the circle as the semi-conjugate axis is to the semi-transverse. Designating by $\pi$ the semi-circumference of the circle whose radius is unity, $\pi \text{A}^2$ will be the area of the circle described upon the

transverse axis. We shall then have for the area of the ellipse,

$$S = \pi . \ AB.$$

### *Of the Parabola.*

216. We have found for the general equation of intersection of the cone and plane, referred to the vertex of the cone (Art. 167),

$$y^2 \sin^2 v + x^2 \sin (v + u) \sin (v - u) - 2cx \sin v \cos v \cos u = o.$$

This equation represents a parabola (Art. ) when $u = v$, which gives

$$y^2 \sin^2 v + 2cx \sin v \cos^2 v = o, \text{ or } y^2 - \frac{2cx \cos^2 v}{\sin v} = o;$$

for the general equation of the parabola referred to its vertex.

Making $y = o$ to find the points in which it cuts the axis, and we have

$$x = o,$$

hence the curve cuts the axis at the origin.

Making $x = o$, determines the points in which it cuts the axis of $y$. This supposition gives

$$y^2 = o,$$

hence the axis of $y$ is tangent to the curve at the origin.

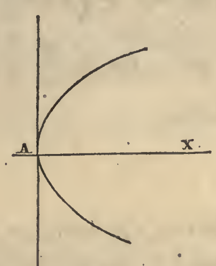
217. Resolving the equation with respect to $y$, we have

$$y = \pm \cos v \sqrt{\frac{2cx}{\sin v}}.$$

These two values being equal and with contrary signs, the curve is symmetrical with respect to the axis of $x$. If we suppose $x$ negative, the values of $y$ become imaginary, since the curve does not extend in the direction of the negative abscissas. For every positive value of $x$, those of $y$ will be real, hence the curve extends indefinitely in this direction.

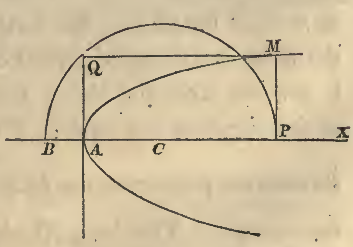
218. The ratio between the square of the ordinate $y^2$ to the abscissa $x$, being the same for every point of the curve, we conclude, *that in the parabola the squares of the ordinates are to each other as the corresponding abscissas.*

219. The line AX is called the *axis* of the parabola, the point A the *vertex*, and the constant quantity $\frac{2c \cos^2 v}{\sin v}$ the *parameter*. For abbreviation make $\frac{2c \cos^2 v}{\sin v} = 2p$, the equation of the parabola becomes




$$y^2 = 2px.$$

220. To describe the parabola, lay off on the axis AX in the direction AB, a distance $AB = 2p$. From any point C taken on the same axis, and with a radius equal to CB, describe a circumference of a circle; from the extremity of its diameter at P, erect the perpendicular PM; and drawing through the point Q, QM parallel to the axis of $x$, the point M will



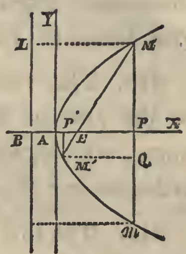
be a point of the parabola. For by this construction we have

$$PM = AQ, \text{ and } \overline{AQ}^2 = AB.\ AP;$$

hence,

$$\overline{MP}^2 = 2p.\ AP.$$

221. If we take on the axis of the parabola the point F at a distance from the vertex equal to $\frac{p}{2}$, we shall have for every point M of the parabola, the relation




$$\overline{FM}^2 = y^2 + \left(x - \frac{p}{2}\right)^2 = 2px + x^2 - px + \frac{p^2}{4} = \left(x + \frac{p}{2}\right)^2;$$

hence,

$$FM = x + \frac{p}{2},$$

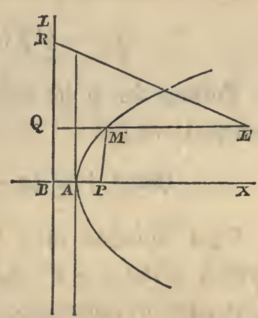
that is, the distance of any point of the parabola from this point is equal to its abscissa, augmented by the distance of the fixed point from the vertex. The point F is called the *focus* of the parabola. Hence we see that in the parabola, as well as the ellipse, the distance of any of its points from the focus is expressed in rational functions of the abscissa. It follows, from the above demonstration, that all the points of the parabola are equally distant from the focus and a line BL drawn parallel to the axis of $y$, and at a distance $\frac{p}{2}$ from the vertex. The line BL is called the *Directrix of the Parabola.*

222. From this property results a second method of describing the parabola when we know its parameter. From

the point A, lay off on both sides of the axis of $y$, distances AB and AF, equal to a fourth of the parameter. Through any point P of the axis erect the perpendicular PM, and from F as a centre with a radius equal to PB, describe an arc of a circle, cutting PM in the two points M M′, these points will be on the parabola. For, from the construction, we have

$$\text{FM} = \text{AP} + \text{AB} = x + \frac{p}{2}.$$

223. The same property enables us to describe the parabola mechanically. For this purpose, apply the triangle EQR to the directrix BL. Take a thread whose length is equal to QE, and fix one of its extremities at E, and the other at the focus F. Press the thread by means of a pencil along the line QE, at the same time slipping the triangle EQR along the directrix, the pencil will describe a parabola. For,




$$\text{FM} + \text{ME} = \text{QM} + \text{ME}, \text{ or QM} = \text{MF}.$$

224. If we make $x = \frac{1}{2}p$ in the equation of the parabola, we get

$$y^2 = p^2, \text{ or } y = p, \text{ or } 2y = 2p.$$

Hence *in the parabola, the double ordinate passing through the focus, is equal to the parameter.*

225. Let it be required to find the equation of a tangent line to the parabola whose equation is

$$y^2 = 2px.$$

Let $x'' \ y''$ be the co-ordinates of the point of tangency,

they must satisfy the equation of the parabola, and we have

$$y''^2 = 2px''.$$

The equation of the tangent line will be of the form

$$y - y'' = a\,(x - x'').$$

It is required to determine $a$.

Let the tangent be regarded as a secant, whose points of intersection coincide. To determine the points of intersection, the three preceding equations must subsist at the same time. Subtracting the second from the first, we have

$$(y - y'')\,(y + y'') = 2p\,(x - x'').$$

Putting for $y$ its value drawn from the equation of the tangent, we get

$$(2ay'' + a^2\,(x - x'') - 2p)\,(x - x'') = o.$$

This equation may be satisfied by making $x - x'' = o$, which gives $x = x''$ and $y = y''$ for the co-ordinates of the first point of intersection, or by making

$$2ay'' + a^2\,(x - x'') - 2p = o.$$

This equation will make known the other value of $x$ when $a$ is known. But when the secant becomes a tangent, the points of intersection unite, and we have for this point also $x = x''$, which reduces the last equation to

$$2ay'' = 2p\,;$$

hence,

$$a = \frac{p}{y''}.$$

Substituting this value in the equation of the tangent, it becomes

$$y - y'' = \frac{p}{y''}(x - x''),$$

or reducing and observing that $y''^2 = 2px''$, we have

$$yy'' = p\,(x + x''),$$

for the equation of the tangent line.

226. By the aid of these formulas we may draw a tangent to the parabola from a point without, whose co-ordinates are $x'$, $y'$. For this point being on the tangent, we have

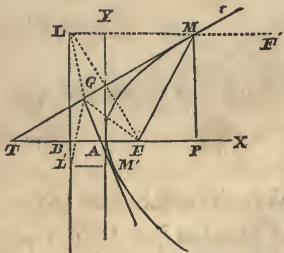
$$y'y'' = p\,(x' + x''),$$

and joining with this the relation

$$y''^2 = 2px'',$$

we may from these equations determine the co-ordinates of the point of tangency. The resulting equation being of the second degree, there may in general be two tangents drawn to the parabola, from a point without.

227. To find the point in which the tangent meets the axis of $x$, make $y = o$ in the equation




$$yy'' = p\,(x + x''),$$

we get

$$x = -\,x'',$$

which is the value of AT. Adding to it the abscissa AP, without regarding the sign, we shall have the subtangent,

$$\mathrm{PT} = 2x'',$$

that is, *in the parabola, the subtangent is double of the abscis-*

*sa*. This furnishes a very simple method of drawing a tangent to the parabola, when we know the abscissa of the point of tangency.

228. The equation of a line passing through the point of tangency is of the form

$$y - y'' = a' (x - x'').$$

In order that this line be perpendicular to the tangent, for which we have (Art. 225),

$$a = \frac{p}{y''},$$

it is necessary that we have

$$aa' + 1 = o,$$

hence

$$a' = -\frac{y''}{p}.$$

The equation of the normal becomes

$$y - y'' = -\frac{y''}{p} (x - x'').$$

Making $y = o$, we have

$$x - x'' = p,$$

which shows that *in the parabola the subnormal is constant and equal to half the parameter.*

229. The directions of the tangent and normal have remarkable relations with those of the lines drawn from the focus to the point of tangency.

The equation of a line passing through the point of tangency is

$$y - y'' = a' (x - x''),$$

and the condition of its passing through the focus, for which $y = o,\ x = \frac{p}{2}$ gives

$$a' = \frac{-y''}{\frac{p}{2} - x''}$$

The angle FMT which this line makes with the tangent, has for a trigonometrical tangent (Art. 64),

$$\frac{a' - a}{1 + aa'}$$

Substituting for $a$ its value $\frac{p}{y''}$, and for $a'$ that found above, and observing that $y''^2 = 2px''$, we have

$$\text{tang FMT} = \frac{p}{y''} = a;$$

hence, *in the parabola, the tangent line makes equal angles with the axis, and with a line drawn from the focus to the point of tangency*, so that the triangle FMT is always isosceles; consequently, when the point of tangency M is given, to draw a tangent, we have only to lay off from F towards T a distance FT = FM. FM will be the tangent required.

230. If through M we draw MF′ parallel to the axis, the tangent will make the same angle with this line as with the axis, hence in the parabola the lines drawn from the point of tangency to the focus and parallel to the axis make equal angles with the tangent. From this, results a very simple

method of drawing a tangent to the parabola from a point without. Let G be the point, F the focus, BL the directrix. From G as a centre, with a radius equal to GF, describe a circumference of a circle, cutting BL, in L, L′. From these points draw LM, L′M′, parallel to the axis. M and M′ will be the points of tangency, and GM, GM′, will be the two tangents that may be drawn from the point G. For, by the nature of the parabola ML = MF, and by construction GF = GL, the line GM has all of its points equally distant from F and L. It is therefore perpendicular to the line FL, consequently the angle LMG, or its opposite *t*MF′, is equal to the angle GMF. MG is therefore a tangent at the point M. The same may be proved with regard to GM′.

### *Of the Parabola referred to its Diameters.*

231. Let us now examine if there are any systems of oblique co-ordinates, relatively to which the equation of the parabola will retain the same form as when it is referred to its axis. The general formulas for transformation are

$$x = a + x' \cos \alpha + y' \cos \alpha', \quad y = b + x' \sin \alpha + y' \sin \alpha'.$$

These values being substituted in the equation of the parabola

$$y^2 = 2px,$$

it becomes

$$\left.\begin{array}{l} y'^2 \sin^2\alpha' + 2x'y' \sin \alpha \sin \alpha' + x'^2 \sin^2\alpha + b^2 - 2ap \\ + 2(b \sin \alpha' - p \cos \alpha')\, y' + 2\,(b \sin \alpha - p \cos \alpha)\, x' \end{array}\right\} = o.$$

In order that this equation preserve the same form as the preceding, we must have

$\sin \alpha' \sin \alpha = o$, $\sin^2\alpha = o$ $b \sin \alpha' - p \cos \alpha' = o$, $b^2 - 2ap = o$,

and the equation reduces to

$$y'^2 = \frac{2p}{\sin^2\alpha'} x' ;$$

and putting for $\frac{p}{\sin^2\alpha'}$, $p'$, we have

$$y'^2 = 2p'x'.$$

232. The second of the preceding equations of conditions shows that $\sin \alpha = o$, that is, the axis of $x'$ is parallel to the axis of $x$. Hence, *all the diameters of the parabola are parallel to the axis.*

233. The two other equations give

$$b^2 = 2ap, \text{ tang } \alpha' = \frac{p}{b}.$$

The first shows that the co-ordinates $a$ and $b$ of the new origin satisfy the equation of the parabola. This origin is therefore a point of the curve. The second determines the inclination of the axis of $y'$ to the axis of $x$, and shows that this axis is tangent to the curve at the origin, since it makes the same axis of $x$ as the tangent line at this point (Art. ).

234. The equation $y'^2 = 2p'x'$, giving two equal values for $y'$, and with contrary signs for each value of $x'$, each diameter bisects the corresponding ordinates.

235. The equation of the parabola being of the same form when referred to its diameters and axis, all of its properties which are independent of the inclination of the co-ordinates will be the same in these two systems. Thus, to

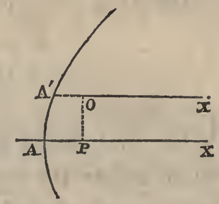
describe a parabola when we know the parameter of one of its diameters, and the inclination of the corresponding ordinates, describe a parabola on this diameter as an axis with the given parameter, and then incline the ordinates without changing their lengths, we shall have the parabola required.

### *Of the Polar Equation of the Parabola, and of the Measure of its Surface.*

236. Let us resume the equation of the parabola referred to its axis,

$$y^2 = 2px,$$

and take O for the position of the pole, the co-ordinates of which are $a$ and $b$; draw OX′ parallel to the axis. The formulas for transformation are (Art. 136),




$$x = a + r \cos v, \quad y = b + r \sin v.$$

Substituting these values in the equation of the parabola, it becomes

$$r^2 \sin^2 v + 2\,(b \sin v - p \cos v)\, r + b^2 - 2pa = o.$$

If the pole be on the curve,

$$b^2 - 2pa = o,$$

and the equation reduce to

$$r^2 \sin^2 v + 2\,(b \sin v - p \cos v)\, r = o,$$

which may be satisfied by making

$$r = o, \text{ or } r \sin v + 2\,(b \sin v - p \cos v) = o.$$

The last equation gives

$$r = \frac{2\,(p \cos v - b \sin v)}{\sin^2 v}.$$

If this second value of $r$ were zero, the radius vector would be tangent to the curve. But this supposition requires that we have

$$2p \cos v - 2b \sin v = o,$$

which gives

$$\frac{\sin v}{\cos v} = \text{tang } v = \frac{p}{b},$$

which is the same value found for the inclination of the tangent to the axis (Art. 225).

237. If the pole be placed at the focus of the parabola, the co-ordinates of which are $b = o \quad a = \frac{p}{2}$, the general polar equation becomes

$$r \sin^2 v - 2p \cos v. \ r = p^2$$

and the values of $r$ are

$$r = \frac{p\,(\cos v + 1)}{\sin^2 v}, \qquad r = \frac{p\,(\cos v - 1)}{\sin^2 v}.$$

The second value of $r$ being always negative, since $\cos v < 1$ and $(\cos v - 1)$ consequently negative, must be rejected. The first value is always positive, and will give real points to the curve. It may be simplified by putting for $\sin^2 v$, $1 - \cos^2 v$, which is equal to $(1 + \cos v)\,(1 - \cos v)$, and this value reduces to

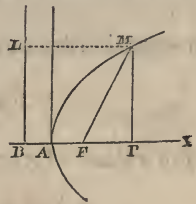
$$r = \frac{p\,(1 + \cos v)}{(1 + \cos v)\,(1 - \cos v)} = \frac{p}{1 - \cos v},$$

which is the polar equation of the parabola when the pole is at the focus.

238. If $v = o$, $r = \frac{p}{o} =$ infinity. Every other value of $v$ from zero to 360° will give finite values to $r$. When $v = 90°$, $\cos v = o$ and $r = p$. When $v = 180°$, $\cos v = -1$ and $r = \frac{p}{2}$, results which correspond with those already found.

239. In the preceding discussion we have deduced all the properties of the parabola from its equation; reciprocally we may find its equation when one of these properties is known.

Let it be required, for example, to find a curve such that the distances of each of its points from a given line and point shall be equal. Let F be the given, BL the given line. Take the line FB perpendicular to BL for the axis of $x$, and place the origin at A, the middle of BF, and make BF $= p$.




For every point M of the curve, we shall have these relations

$$\overline{\mathrm{FM}}^2 = y^2 + \left(x - \frac{p}{2}\right)^2,$$

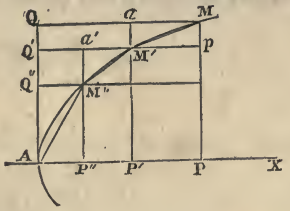
$$\mathrm{FM} = x + \frac{p}{2};$$

eliminating $r$, we have

$$y^2 = 2px,$$

which is the equation of the parabola.

To find the area of any portion of the parabola, let APM be the parabolic segment whose area is required. Draw MQ parallel and AQ perpendicular to the axis. The area of the segment APM is two-thirds of the rectangle APQM.




Inscribe in the parabola any rectilinear polygon MM′M″. From the vertices of this polygon draw parallels to the lines AP, PM, forming the interior rectangles PP′pM, P′P″p′M″, and the corresponding exterior rectangles QQ′q′M″. Representing the first by P, P′, P″ and the last $p$, $p'$, $p''$, we shall have

$$\mathrm{P} = y'\,(x - x'), \quad p = x'\,(y - y'),$$

which gives

$$\frac{\mathrm{P}}{p} = \frac{y'\,(x - x')}{x'\,(y - y')};$$

but the points M, M′, M″, belong to the parabola, and we have

$$y^2 = 2px, \quad y'^2 = 2px',$$

which gives

$$(x - x') = \frac{y^2 - y'^2}{2p}, \quad x' = \frac{y'^2}{2p}.$$

Substituting these values, the ratio of P to $p$ becomes

$$\frac{\mathrm{P}}{p} = \frac{y'\,(y^2 - y'^2)}{y'^2\,(y - y')} = \frac{y + y'}{y'}.$$

The same reasoning will apply to all the sides of the polygon, and we have the equations

$$\frac{P}{p} = \frac{y + y'}{y'}$$

$$\frac{P'}{p'} = \frac{y' + y''}{y''}$$

$$\frac{P''}{p''} = \frac{y'' + y'''}{y'''}, \text{ \&c.}$$

The polygon M, M′, M″, being entirely arbitrary, we may place the vertices in such a manner that designating by $u$ any constant quantity, we have always

$$y - y' = uy'$$

$$y' - y'' = uy''$$

$$y'' - y''' = uy''', \text{ \&c.}$$

which is equivalent to making $y$, $y'$, $y''$, decrease in a geometrical progression. But from supposition we have

$$\frac{y + y'}{y'} = 2 + u$$

$$\frac{y' + y''}{y''} = 2 + u$$

$$\frac{y'' + y'''}{y'''} = 2 + u, \text{ \&c.}$$

and the several ratios become

$$\frac{P}{p} = 2 + u,$$

$$\frac{P'}{p'} = 2 + u,$$

$$\frac{P''}{p''} = 2 + u, \text{ \&c.}$$

Hence these ratios will be equal, whatever be $u$. By composition we have

$$\frac{P + P' + P'' + \&c.}{p + p' + p''\ldots\ldots} = 2 + u;$$

but the numerator of the first member is the sum of the interior rectangles, and the denominator that of the exterior rectangles. As $u$ diminishes, this ratio approaches more and more the value of Q, and we may take $u$ so small, that the difference will be less than any assignable quantity. But, under this supposition, the inscribed and circumscribed rectangles approach a coincidence with the inscribed and circumscribed cuvilinear segments, consequently the limit of their ratio is equal to the ratio of the segments, and representing the first by S, and the second by $s$, we have

$$\frac{S}{s} = 2,$$

which gives

$$\frac{S + s}{s} = 3,$$

and dividing these equations number by number,

$$S = \frac{2}{3}(S + s);$$

but $S + s$ is the sum of the inscribed and circumscribed segments, and is consequently the surface of the rectangle APMQ. *Hence, the area of the parabolic segment* APM *is equal to two-thirds of the rectangle described upon its abscissa and ordinate.*

240. *Quadrable Curves* are those curves any portion of

whose area may be expressed in a finite number of algebraic terms. The parabola is quadrable, while the ellipse is not.

## *Of the Hyperbola.*

241. We have found (Art. 167) for the general equation of the conic sections,

$$y^2 \sin^2 v + x^2 \sin (v + u) \sin (v - u) - 2cx \sin v \cos v \cos u = o,$$

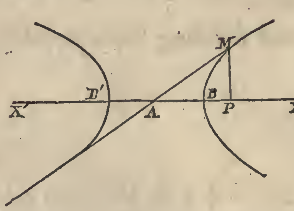
and (Art. 146) that this equation represents a class of curves called *Hyperbolas*, when $u > v$.

To discuss this curve, let us find the points in which it cuts the axis of $x$, make $y = o$, we have

$$x^2 \sin (v + u) \sin (v - u) - 2cx \sin v \cos v \cos u = o,$$

which gives for the two values of $x$

$$x = o,\ x = \frac{2c \sin v \cos v \cos u}{\sin (v + u) \sin (v - u)},$$




which show that the curve cuts this axis at two points B B′, one of which is at the origin, and the other at a distance $BB' = \dfrac{2c \sin v \cos v \cos u}{\sin(v+u)\sin(v-u)}$ from the origin, and on the negative side of the axis of $y$, since $\sin (v - u)$ is negative. Making $x = o$, we find

$$y^2 = o;$$

hence the axis of $y$ is tangent to the curve at the origin.

242. Resolving this equation with respect to $y$, we have

$$y = \frac{1}{\sin v}\sqrt{-x^2 \sin(v+u)\sin(v-u) + 2cx \sin v \cos v \cos u}.$$

These two values being equal, and with contrary signs, the curve is symmetrical with respect to the axis of $x$. For every positive value of $x$, we shall have a *real* value of $y$, since $\sin(v-u)$ being negative, the sign of the first term under the radical is essentially positive. The curve therefore extends indefinitely on the positive side of the axis of $y$. If $x$ be negative, $y$ will only have real values when $-x^2 \sin(v+u)\sin(v-u) > 2cx \sin v \cos v \cos u$. Putting the value of $y$ under the form

$$y = \frac{1}{\sin v}\sqrt{-x\sin(v+u)\sin(v-u)\left(x - \frac{2c \sin v \cos v \cos u}{\sin(v+u)\sin(v-u)}\right)},$$

we see that the factor $-x\sin(v+u)\sin(v-u)$ will be always positive, whatever be the sign of $x$. The value of $y$ will therefore be real so long as the negative value attributed to $x < \frac{2c \cos v \cos u \sin v}{\sin(v+u)\sin(v-u)}$, and that they will be imaginary for every negative value of $x$ from

$$x = o \text{ to } x = \frac{2c \sin v \cos v \cos u}{\sin(v+u)\sin(v-u)},$$

that is, from B to B′; hence there is no point of these points, but it extends indefinitely from B′ in the direction BB′.

243. Let the origin of co-ordinates be taken at A, the middle of BB′.

The formula for transformation is

$$x = x' + \frac{c \sin v \cos v \cos u}{\sin (v + u) \sin (v - u)}.$$

Substituting this value of $x$ in the equation of the curve, and reducing, we have

$$y^2 \sin^2 v + x'^2 \sin (v + u) \sin (v - u) + \frac{c^2 \sin^2 v \cos^2 v \cos^2 u}{\sin^2 (v + u) \sin^2 (v - u)} = o.$$

Making $y = o$, to find the point in which it cuts the axis of $x$, we find

$$x' = \text{AB} = \pm \frac{c \sin v \cos v \cos u}{\sin (v + u) \sin (v - u)};$$

but for $x = o$, we find that the values of $y$ are imaginary; the curve therefore does not intersect the axis of $y$.

If we make

$$\text{A}^2 = \frac{c^2 \sin^2 v \cos^2 v \cos^2 u}{\sin^2 (v + u) \sin^2 (v - u)}, \text{ and}$$

$$\text{B}^2 = - \frac{c^2 \cos^2 v \cos^2 u}{\sin (v + u) \sin (v - u)},$$

and multiplying the two members of the equation of the highest value by

$$\frac{c^2 \cos^2 v \cos^2 u}{\sin^2 (v + u) \sin^2 (v - u)},$$

and put $x$ for $x'$, we shall have

$$\text{A}^2 y^2 - \text{B}^2 x^2 = - \text{A}^2 \text{B}^2$$

for the equation of the hyperbola referred to its *centre and axes.*

244. The quantities 2A, 2B, are called the *axes* of the hyperbola. The point A is the centre. Every line drawn through the centre and terminated in the curve is called a *diameter*, and there results from the symmetrical form of the hyperbola that every diameter is bisected at the centre.

245. The equation of the ellipse referred to its centre and axes, is

$$A^2y^2 + B^2x^2 = A^2B^2.$$

Comparing this equation with that of the hyperbola, we see *that to pass from one to the other we have only to change* B *into* B $\sqrt{-1}$. This simple analogy is important from the facility it affords in passing from the properties of the ellipse to those of the hyperbola.

246. When the two axes of the hyperbola are equal, its equation becomes

$$y^2 - x^2 = - A^2;$$

we say then that the hyperbola is *equivalateral.*

When the axes of the ellipse are equal, its equation becomes

$$y^2 + x^2 = A^2,$$

which is the equation of a circle. The equilateral hyperbola is then to the common hyperbola what the circle is to the ellipse.

247. It follows from this analogy between the ellipse and hyperbola, that if these curves have equal axes and are placed one upon the other, the ellipse will be comprehended within the limits, between which the hyperbola becomes imaginary, and reciprocally, the hyperbola will have real ordinates, when those of the ellipse are imaginary.

248. The equation of a line passing through the point B′, for which $y = o$, $x = - A$, is

$$y = a\,(x + A).$$

That of a line passing through B, for which $y = o$, $x = +$ A, is

$$y = a\,(x - A).$$

In order that these lines intersect on the hyperbola, these equations must subsist at the same time with that of the hyperbola. Multiplying them member by member, we have

$$y^2 = aa'\,(x^2 + A^2).$$

Combining this with the equation of the hyperbola, put under the form

$$y^2 = \frac{B^2}{A^2}\,(x^2 - A^2),$$

we have

$$aa' = \frac{B^2}{A^2},$$

which establishes a constant relation between the angles which the supplementary chords make with the axis of $x$.

249. When the hyperbola is equilateral $B = A$, and this relation reduces to

$$aa' = 1,$$

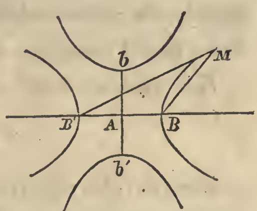
which shows that in the equilateral hyperbola, *the sum of the two acute angles which the supplementary chords make*

*with the transverse axis, on the same side, is equal to a right angle.*

250. If we put $x$ in the place of $y$ and $y$ for $x$ in the equation of the hyperbola, it becomes

$$B^2y^2 - A^2x^2 = A^2B^2.$$

If in this equation we make $x = o$, $y$ becomes real, and $y = o$ makes $x$ imaginary. Hence the curve cuts the axis of $y$, but does not meet that of $x$. It is then situated as in the figure, the transverse axis being $b$, $b'$. The curve is said to be referred to its conjugate axis, because the abscissas are reckoned on this axis.




251. The analogy between the ellipse and hyperbola leads us to inquire if there are not points in the hyperbola corresponding to the foci of the ellipse.

In the ellipse the abscissas of these points were

$$x = \pm \sqrt{A^2 - B^2}.$$

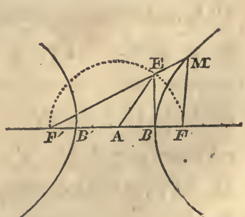
Changing B into B $\sqrt{-1}$, we have for the hyperbola

$$x = \pm \sqrt{A^2 + B^2}.$$

Let us for simplicity make

$$c = \pm \sqrt{A^2 + B^2},$$

and let F, F′, be the points at this distance from the centre, we will have




$$FM^2 = y^2 + (x - c)^2 = \frac{B^2}{A^2}(x^2 - A^2) + x^2 - 2cx + c^2,$$

from which we obtain

$$\mathrm{FM} = \frac{cx}{\mathrm{A}} - \mathrm{A}.$$

In the same manner we will have

$$\mathrm{F'M} = \frac{cx}{\mathrm{A}} + \mathrm{A},$$

that is, *the distances* FM, F'M, *are expressed in rational functions of the abscissa* $x$.

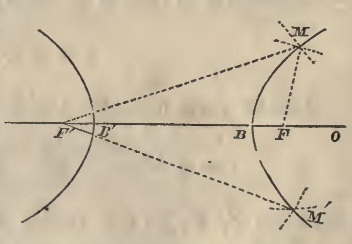
Subtracting these equations from each other, we get

$$\mathrm{F'M'} - \mathrm{FM} = 2\mathrm{A}.$$

Hence, *the difference of these distances is constant and equal to the transverse axis.*

252. To find the position of the foci geometrically, erect at one of the extremities of the transverse axis a perpendicular BE = B the semi-conjugate axis, and draw AE. From the point A as a centre with a radius AE describe a circumference of a circle, cutting the axis in F, F'. These points are the foci of the hyperbola.

253. The preceding properties enable us to construct the hyperbola. From the focus F as a centre with a radius BO, describe a circumference of a circle. From F' as a centre with a radius B'O = BB' + BO describe another circumference. The points M, M', in which they intersect, are points of the hyperbola, for




$$\mathrm{F'M} - \mathrm{FM} = 2\mathrm{A}.$$

254. By following the same course explained in Art. 182,

for the ellipse, we may find the equation of a tangent line to the hyperbola. But this equation may be at once obtained by making $B = B\sqrt{-1}$ in the equation of a tangent line to the ellipse, and have

$$A^2yy'' - B^2xx'' = -A^2B^2$$

for the equation of a tangent line to the hyperbola.

255. The equation of a line passing through the centre and point of tangency is

$$y'' = a'x'',$$

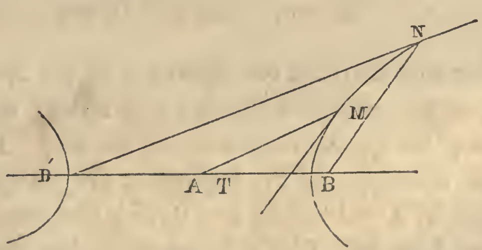
which gives

$$a' = \frac{y''}{x''}.$$

Multiplying this by the value of $a$ corresponding to the tangent, we have

$$aa' = \frac{B^2}{A^2}.$$

Comparing this result with Art. 248, we find the same value for $aa'$. Hence to draw a tangent to the hyperbola, at any




point M, draw the diameter AM, then through B′ draw the chord B′N parallel to AM; MT parallel to BN will be the tangent required.

*Of the Hyperbola referred to its Conjugate Diameters.*

256. The properties of the hyperbola referred to its diameters may be easily deduced from those of the ellipse. By making $B' = B'\sqrt{-1}$ in the equation of the ellipse (Art. 201), we find

$$A'^2y^2 - B'^2x'^2 = - A'^2B'^2.$$

The quantities $2A'$, $2B'$, are called the conjugate diameters of the hyperbola.

This equation could be also obtained by the same method demonstrated for finding the equation of the ellipse.

257. In the same manner, by making $B = B\sqrt{-1}$, and $B' = B\sqrt{-1}$ in the equations Art. 205, we have the relation

$$A'^2 - B'^2 = A^2 - B^2,$$

$$A'B' \sin(\alpha' - \alpha) = AB,$$

$$A^2 \text{ tang } \alpha \text{ tang } \alpha' = B^2 = o.$$

The first signifies that *the difference of the squares constructed on the conjugate diameters is always equal to the difference of the squares constructed on the axes.* Hence the conjugate diameters of the hyperbola are unequal. The supposition of $A' = B'$ gives $A = B$, and reciprocally. *The equilateral hyperbola is the only one which has equal conjugate diameters.*

The second of the preceding equations shows *that the parallelogram constructed on the conjugate diameters is always equivalent to the rectangle on the axes.*

The third relation compared with that of Art. 248, shows *that the supplementary chords drawn to the transverse axis are respectively parallel to two conjugate diameters.*

*Of the Asymptotes of the Hyperbola, and of the Properties of the Hyperbola referred to its Asymptotes.*

258. The indefinite extension of the branches of the hyperbola introduces a very remarkable law which is peculiar to it. The equation of the hyperbola referred to its centre and axes may be put under the form

$$y^2 = \frac{B^2}{A^2}(x^2 - A^2),$$

which gives for the two values of $y$,

$$y = \pm \frac{Bx}{A}\left(1 - \frac{A^2}{x^2}\right)^{\frac{1}{2}}.$$

Developing the second member, it becomes

$$1 - \tfrac{1}{2}\frac{A^2}{x^2} - \tfrac{1}{8}\frac{A^4}{x^4} - \tfrac{1}{16}\frac{A^6}{x^6}, \text{ \&c.};$$

and multiplying by $\frac{Bx}{A}$, it becomes

$$y = \pm\left(\frac{Bx}{A} - \tfrac{1}{2}\frac{BA}{x} - \tfrac{1}{8}\frac{BA^3}{x^3} - \tfrac{1}{16}\frac{BA^5}{x^5}, \text{ \&c.}\right)$$

In proportion as $x$ augments A, and B remaining constant, the terms $\frac{BA}{x}$, $\frac{BA^3}{x^3}$, &c., will diminish. The values of $y$ will continually approach to those of the first term $\pm\frac{Bx}{A}$. As $x$

is indefinite, we may give it such a value as to make the difference smaller than any assignable quantity. If, therefore, we construct the two lines whose equations are represented by

$$y = + \frac{Bx}{A}, \quad y = - \frac{Bx}{A},$$

these lines will be the limits of the branches of the hyperbola, which they will continually approach without ever meeting. And this may be readily shown, for we have

$$y^2 = \frac{B^2x^2}{A^2} - B^2 \text{ for points on the hyperbola;}$$

$$y^2 = \frac{B^2x^2}{A^2} \text{ for points on the lines;}$$

which shows that the ordinates corresponding to the same abscissas are always smaller for the curve than for the lines. These lines are called *Asymptotes.*

259. We can easily prove from the preceding expressions that the asymptotes continually approach the hyperbola; for, subtracting the first from the second, and designating the ordinates of the asymptotes by $y'$, we have

$$y'^2 - y^2 = B^2,$$

or,

$$(y' - y)\,(y' + y) = B^2;$$

hence,

$$y' - y = \frac{B^2}{y' + y};$$

$y' - y$ is the difference of the ordinates of the asymptotes and hyperbola. The fraction which expresses this value has a constant numerator, while the denominator varies with $y'$

and $y'$. The more $y$ and $y'$ increase, the smaller will be this difference. As there is no limit to the values of $y$ and $y'$, the difference may be made smaller than any assignable quantity.

260. To construct the asymptotes of the hyperbola, draw through the extremity of the transverse axis a perpendicular, on which lay off above and below the axis of $x$ two distances equal to half of the conjugate axis. Through the centre of the hyperbola and the extremities of these distances, draw two lines; they will be the asymptotes required, for they make with the axis of $x$, angles whose trigonometrical tangents are $\pm \frac{B}{A}$.

261. If the hyperbola be equilateral, $B = A$, and the asymptotes make angles of 45° with the axis of $x$.

262. The asymptotes are the limits of all tangents drawn to the hyperbola. In fact, the equation of a tangent line to this curve being (Art. 254),

$$A^2yy'' - B^2xx'' = - A^2 B^2,$$

the point in which it meets the axis of the hyperbola, has for an abscissa

$$x = \frac{A^2}{x''}.$$

In proportion as $x''$, which is the abscissa of the point of tangency, increases, the value of $x$ diminishes; and when $x'' =$ infinity, $x = o$. In this supposition the value of $y''$ becomes also infinite and equal to $\pm \frac{Bx''}{A}$, so that, substituting this value in the expression for $a$, which is $\frac{B^2x''}{A^2y''}$, we find

$$a = \pm \frac{B}{A},$$

which is the value of $a$, corresponding to the asymptotes.

263. The equation of the hyperbola takes a remarkable form when we refer it to the asymptotes as axes. The general formulas for transformation are

$$x = x' \cos \alpha + y' \cos \alpha', \quad y = x' \sin \alpha + y' \sin \alpha'.$$

But, as the asymptotes make with the axis of $x$ angles whose tangents are $\pm \frac{B}{A}$, we have

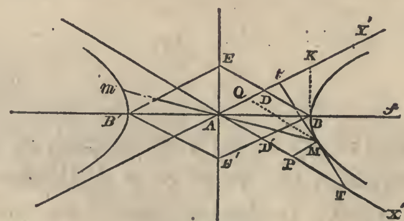
$$\text{tang}\, \alpha = -\frac{B}{A}, \quad \text{tang}\, \alpha' = +\frac{B}{A}.$$

Substituting these values of $x$ and $y$ in the equation of the hyperbola,

$$A^2y^2 - B^2x^2 = -A^2 B^2,$$

it becomes

$$\left.\begin{array}{l}(A^2 \sin^2\alpha' - B^2 \cos^2\alpha')\, y'^2 + (A^2 \sin^2\alpha - B^2 \cos^2\alpha) \\ \quad x'^2 + 2(A^2 \sin \alpha \sin \alpha' - B^2 \cos \alpha \cos \alpha')\, x'\, y'\end{array}\right\} = -A^2 B^2.$$




The coefficients of $x'^2$, $y'^2$, disappear in virtue of the preceding value of tang $\alpha$, tang $\alpha'$, and that of $x'\, y'$ reduces to $-\frac{4\, A^2\, B^2}{A^2 + B^2}$, and the equation of the curve becomes

$$x'y' = \frac{A^2 + B^2}{4}.$$

If we take the line BB′ for the transverse axis of the hyperbola, and AX′, AY′, for the asymptotes, BE parallel to AX′ will be equal to $\sqrt{A^2 + B^2}$. But BK drawn perpendicular to BB′ at B is equal to BE. Hence, AK = BE, and AD = BD. As the same thing may be shown with respect to the other asymptote, ADBD′ will be a rhombus, whose side AD $= \frac{1}{2}$AK $= \frac{\sqrt{A^2 + B^2}}{4}$. Let B represent the angle X′AY′ which the asymptotes make with each other, the preceding equation of the hyperbola multiplied by sin B, gives

$$x'y' \sin B = \frac{A^2 + B^2}{4} \sin B.$$

The first member represents the area of the parallelogram APMQ, constructed upon the co-ordinates AP, PM, of any point of the hyperbola; the second member represents the area of the parallelogram ADBD′, constructed upon the co-ordinates AD′, D′B, of the vertex B of the hyperbola. Hence the area APMQ is equivalent to that of the figure ADBD′. The rhombus BEB′E is called *the Power of the Hyperbola.*

264. When the hyperbola is equilateral A = B, angle B = 90°, sin B = 1 and the rhombus ADBD′ becomes a square which is equivalent to the rectangle of the co-ordinates. For more simplicity, put $\frac{A^2 + B^2}{4} = M^2$, and suppress the accents of $x'$, $y'$, we shall have

$$xy = M^2,$$

for the equations of the hyperbola referred to its asymptotes.

265. By pursuing the same method which has been explained, we may find the equation of a tangent line to the hyperbola referred to its asymptotes. This equation is

$$y - y'' = -\frac{y''}{x''}(x - x'').$$

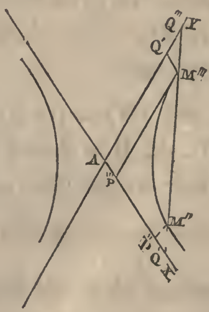
Making $y = o$ gives the point in which it cuts the axis of $x$, and $x - x''$ will be the subtangent, which we find to be

$$x - x'' = x'',$$

that is, the subtangent is equal to the abscissa of the points of tangency. To draw the tangent, take on the asymptote a length PT = AP = $x''$, MT will be the tangent required. We see by this construction, that if we produce the line MT until it meets the other asymptote at $t$, we shall have M$t$ = MT. The portion of the tangent which is comprehended between the asymptotes is therefore bisected at the point of tangency.

266. The equation of a line passing through any point M'', whose co-ordinates are $x''$, $y''$, is

$$y - y'' = a\ (x - x'').$$




The other point M''' in which this line meets the curve, is determined from the equation

$$ax + y'' = o,$$

which gives

$$x = -\frac{y''}{a}.$$

This is the value of the abscissa AP‴. But if we make $y=o$ in the equation of the straight line, it gives also

$$x - x'' = -\frac{y''}{a},$$

in which $x$ represents the abscissa AQ″ of the point in which this line meets the axis A$x$, and $x - x''$ is the value of P″Q″. Hence P″Q″ = AP‴. Consequently if we draw M‴Q′ parallel to AX, the triangles P″M″Q″, Q′M‴Q‴ will be equal, and the lines M″Q″, M‴Q‴, will be also equal, that is, *if through any point of the hyperbola, a straight line be drawn terminated in the asymptotes, the positions of this line comprehended between the asymptotes and the curve will be equal.*

267. This furnishes us with a very simple method of describing the hyperbola by points, when we know one point M″ and the position of its asymptotes, for drawing through this point any line Q″M″Q‴ terminated by the asymptotes, and laying off from Q‴ to M‴ the distance Q″M″, M″ will be a point of the curve. Drawing any other line through either of these points, we may in the same way find other points of the curve. This construction may also be used when we know the centre and axes of the hyperbola. For with these given, we may easily construct the asymptotes.

### *Of the Polar Equation of the Hyperbola, and of the Measure of its Surface.*

268. Resuming the equation of the hyperbola referred to its centre and axes,

$$A^2y^2 - B^2x^2 = - A^2B^2,$$

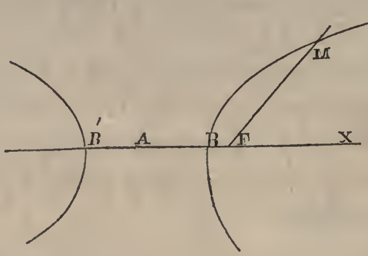
we derive its polar equation, by substituting this value of $a$ and $b$, drawn from the formulas for transformation from rectangular to polar co-ordinates, in the place of $x$ and $y$ in this equation. The substitution gives

$$\left.\begin{array}{l} A^2 \sin^2 v \\ - B^2 \cos^2 v \end{array}\right\} r^2 + \left.\begin{array}{l} 2A^2b \sin v \\ - 2B^2a \cos v \end{array}\right\} r + A^2b - B^2a + A^2B^2 = o$$

for the general polar equation of the hyperbola.

269. When the pole is at one of the foci, we have $a = \pm \sqrt{A^2+B^2}$. $b = o$ taking the positive value of $a$, corresponding to the point F, the substitution gives for the two values of $r$,

$$r = \frac{B^2}{A - a \cos v}, \quad r = -\frac{B^2}{A + a \cos v}.$$




If we make $v = o$, the radius vector takes the position FX. $\cos v = 1$, the denominator of $r$ becomes $A - a = A - \sqrt{A^2 + B^2}$, a quantity which is essentially negative. Hence the curve has no real points in this direction, and this will be the case until $\cos v$ is so small, that the product $a \cos v$ shall be less than A. The condition will be fulfilled when $A + a \cos v = o$, which gives

$$\cos v = \frac{A}{a} = \frac{A}{\sqrt{A^2+B^2}}.$$

This value of the angle $v$ is the same which the asymptotes make with the axis. For every value of $v$ greater than this limit, but less than $90°$, $a \cos v$ is positive, and less than A; when $v > 90°$, $a \cos v$ becomes negative, and $- a \cos v$ positive. In this case $A - a \cos v$ is positive as well as $r$. The points which this value of $r$ gives, correspond then to the branch of the hyperbola situated on the positive side of the axis of $x$.

270. But in discussing the second root, we shall see that it belongs to the other branch. In fact, it gives imaginary values for all values of the cos $v$ between the limits $\cos v = 1$ and $\cos v = -\frac{A}{a}$. All the other values of $v$ greater than that of the second limit will give positive values for $r$, and when $v = 180°$, the radius vector will determine the vertex B′.

271. To put the preceding expressions under the form adopted in the ellipse, make

$$e = \frac{a}{A}, \text{ or } e = \frac{\sqrt{A^2 + B^2}}{A};$$

in which $e$ represents the ratio of the eccentricity to the semi-transverse axis, and the values of $v$ become

$$r = -\frac{A(1 - e^2)}{1 - e \cos v}, \quad r = +\frac{A(1 - e^2)}{1 + e \cos v}.$$

These two equations determine points situated on the two branches of the hyperbola.

272. We have seen that a similar transformation gives two values for the radius vector in the ellipse, but that one of these values is constantly negative and consequently belongs to no point of the curve, while for the hyperbola we

find two separate and rational values for $r$ corresponding to the two branches of the hyperbola. Let us examine this difference. If in the first of the preceding equations, we count the angle $v$ from the vertex of the curve, it will be necessary to change $v$ into $180° - v$, and we have then

$$r = -\frac{A(1 - e^2)}{1 + e \cos v}.$$

This value of $r$ will equally give every point of the branch to which it belongs by attributing suitable angles to $v$. But operating in the same way in Art. 212 on the ellipse, that is, counting the angle $v$ from the nearest vertex, we get

$$r = \frac{A(1 - e^2)}{1 + e \cos v}.$$

This equation is therefore absolutely the same for the two cases, only in the ellipse $e$ is less than unity, while it is greater in the hyperbola. Besides, the sign of A is changed. Let us now make $e = 1$ and A = infinity, we shall have, making $A(1 - e^2) = p$,

$$r = \frac{p}{1 + \cos v},$$

which is the polar equation of the parabola. Hence we see that the equation

$$r = \frac{A(1 - e^2)}{1 + e \cos v},$$

may in general represent all the conic sections, by giving suitable values to A and $e$.

273. We may deduce the equation of the hyperbola in the same manner as we have that of the ellipse in Art.  by introducing one of its properties which characterize it. The

method being similar to that of the ellipse, it will be unnecessary to repeat it here.

274. We have seen that the equilateral hyperbola bears the same relation to other hyperbolas that the circle does to the ellipse. In applying here what has been said (Art. 215), we may compare a portion of any hyperbola, to the corresponding area of an equilateral hyperbola having the same transverse axis, and there results that these are to each other in the ratio of the conjugate axes. The absolute areas however can only be obtained by means of logarithms.

275. We have found (Art. 173) for the equation of the Ellipse referred to its vertex,

$$y^2 = \frac{B^2}{A^2}(2Ax - x^2) ;$$

for the equation of the parabola, we have

$$y^2 = 2px,$$

and for the hyperbola

$$y^2 = \frac{B^2}{A^2}(2Ax + x^2).$$

These equations may all be put under the form

$$y^2 = mx + nx^2,$$

in which $m$ is the paramater of the curve, and $n$ the square of the ratio of the semi-axes.

In the ellipse $n$ is negative, in the hyperbola it is positive, and in the parabola it is *zero*.

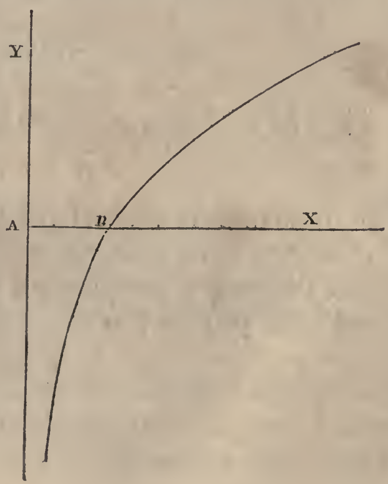
# CHAPTER V.

## OF TRANSCENDENTAL CURVES.

276. Curves whose equations are not purely algebraic, are called *Transcendental.* Their equations involve, generally, logarithmic expressions, and functions of the sine, cosine, tangents, &c., of arcs. As many of these curves enjoy remarkable mechanical properties, we propose discussing a few of those which are most commonly found in works on Mechanics.

### *Of the Logarithmic Curve.*

277. This curve derives its name from one of the co-ordinates being the logarithm of the other.




If the axis of $x$ be taken as the *axis of numbers*, that of $y$ will be the *axis of logarithms;* and laying off any numbers, 1, 2, 3, 4, &c., on AX, the logarithms of these numbers, as found in the Tables of Logarithms, estimated in parallels to the axis of $y$, will be the corresponding ordinates of the curve.

278. From what has been said, the equation of the curve is

$$y = lx;$$

or, calling $a$ the base of the system of logarithms, we have

$$x = a^y.$$

If the base of the system be changed, the values of $y$ will vary for the same value of $x$; hence, *every system of logarithms will produce a different logarithmic curve.*

279. The equation

$$x = a^y$$

enables us at once to construct points of the curve; for, making successively

$$y = o, \quad y = \tfrac{1}{2}, \quad y = \tfrac{3}{4}, \quad \&c.,$$

we find

$$x = 1, \quad x = \sqrt{a}, \quad x = \sqrt{a}, \quad \&c.$$

As $y = o$, gives $x = 1$, whatever be the system of logarithms, it follows that *every logarithmic curve cuts the axis of numbers of an unit's distance from the origin.*

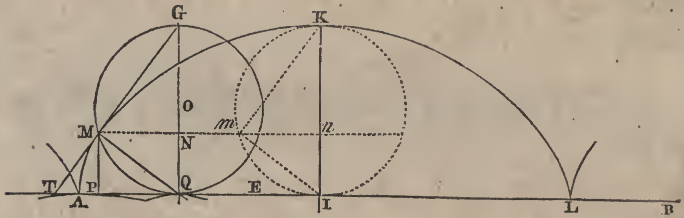
280. If $a > 1$, all values of $x$ greater than unity will give real and positive values for $y$; the curve, therefore, extends indefinitely above the axis of numbers. For values of $x$ less than unity, $y$ becomes negative, and increases as $x$ diminishes; and when $x = o$, $y = -$ infinity. The curve, therefore, extends indefinitely below the axis of numbers, and as it approaches continually the axis of logarithms, this axis *is an asymptote* to the curve.

If $x$ be negative, $y$ becomes imaginary; the curve is, therefore, limited by the axis of logarithms.

281. Taking the axis of $y$ for the axis of numbers, that of $x$ would be the axis of logarithms, and the curve would enjoy, relatively to this system, the same properties which have demonstrated above.

## *Of the Cycloid.*

282. If a circle QMG be rolled along the line AB, any point M of its circumference will describe a curve AMKL, which is called a *Cycloid*. For any position of the genera-




ting circle, as QMG, the distance AQ = arc MQ. Erect the perpendicular QG; it will be a diameter of the circle. Draw MN parallel to AB, MN will be the sine of the arc MQ, and NQ its versed sine. Making

$$\text{QO} = a,\ \text{AP} = x,\ \text{MP} = \text{NQ} = y,$$

we shall have

$$x = \text{AQ} - \text{PQ} = \text{arc MQ} - \sin \text{MQ},\ y = \text{ver-sin MQ};$$

and since MN is a mean proportional between the segments QN and NG, we have

$$MN = \sin MQ = \sqrt{2ay - y^2};$$

hence

$$x = \text{arc } MQ - \sqrt{2ay - y^2},$$

or representing by vers-sin $^{-1} y$ the arc of which $y$ is the versed sine, we have

$$x = \text{versed sine}^{-1} y - \sqrt{2ay - y^2},$$

for the *transcendental equation of the cycloid.*

283. When the point of contact is at a distance AI from the origin equal to the semi-circumference of the generating circle, the generating point is at K, the distance KI being equal to the diameter of the circle; when the circle has made an entire revolution, the generating point is at L. The cycloid is not terminated at this point, but as the generating circle moves on, similar cycloids are described along AB produced.

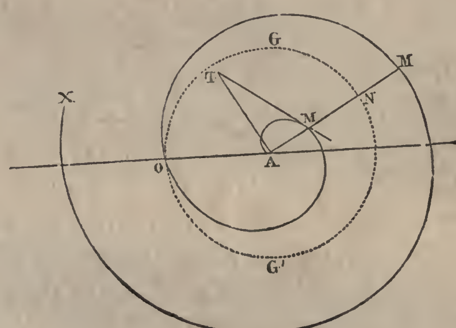
284. AB is called the *base* and KI the *altitude* of the cycloid. This curve enjoys many important mechanical properties. It is a *tautocronal* and *brachystochronal* curve, or curve of equal and swiftest descent.

## *Of Spirals.*

285. Spirals comprise a class of transcendental curves, which are remarkable for their form and properties. The principal varieties are the Spiral of Archimedes, the Hyperbolic, Parabolic and Logarithmic Spiral.

*Spiral of Archimedes.*

286. If a line AO revolve uniformly around a centre A, at the same time that one of its points commencing at A, with a regular angular and outward motion, describes a curve AMO, and is found at O, when AO has completed one




entire revolution, and at X at the end of the second revolution, and so on, the curve AMOM′X will be the spiral of Archimedes. From the nature of this generation, it follows that the ratio of the distance of each of its points from the point A to the line AO, will be equal to that of the arc passed over by the point O from the commencement of the revolution to the entire circumference; or, for any point M′, we have the equation

$$\frac{AM'}{AN} = \frac{OGO + OGN}{OGO},$$

and making

$$OGN = t, \ AM = u, \ AN = 1,$$

the circumference OGO will be represented by $2\pi$, and the equation of the spiral becomes

$$u = \frac{t}{2\pi}.$$

287. The variables in this equation are those of polar co-ordinates. The point A is the *pole*, AM the *radius vector*, and the angle subtending OGN the *variable angle.*

288. The curve which has just been considered is a particular case of the class of spirals, whose general equation may be represented by

$$u = at^n;$$

$a$ and $n$ representing any quantities whatsoever.

### *Of the Hyperbolic Spiral.*

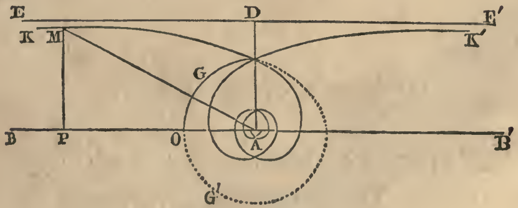
289. If in the general equation

$$u = at^n,$$

we have $n = -1$, the resulting equation

$$u = \frac{a}{t},$$

will be that of the *hyperbolic spiral.* This curve takes its



name from its having an asymptote. In fact, if we make successively

$$t = 1 = \tfrac{1}{2} = \tfrac{1}{3}, \&c.$$

we shall have

$$u = a = 2a = 3a, \&c.$$

which shows that as the spiral departs from the point A, it approaches continually the line DE drawn parallel to AO, and a distance $AD = a$; DE is therefore an asymptote to the curve.

If the values of $t$ be negative, we shall have a similar spiral, to which DE′ will be an asymptote.

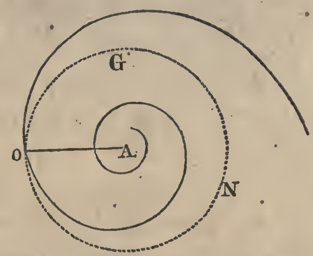
### *Of the Parabolic Spiral.*

290. The parabolic spiral is generated by wrapping the axis of a parabola around the circumference of a circle, the ordinates of the parabola will then coincide with the prolongation of the radii, and the abscissas of the parabola will become the arcs of the circle. The equation of the curve is evidently

$$u^2 = at.$$

### *Of the Logarithmic Spiral.*

291. The equation of this curve is $t = lu$. Making $t = o$, we have $u = 1$. The curve therefore passes through the




point O. As $u$ increases, $t$ increases also; there is therefore an infinite number of revolutions about the circle OGN. When $u < 1$, $t$ becomes negative, and its values give the part of the curve within the circle OGN. As $u$ diminish, $t$ increases, and when $u = o$, $t = -$ infinity. The spiral therefore continually approaches the pole, but never meets it.

# CHAPTER VI.

## DISCUSSION OF EQUATIONS.

292. Having discussed in detail the particular equations of the Circle, Ellipse, Parabola, and Hyperbola, we will apply the principles which have been established to the discussion of the general equation of the second degree between two indeterminates.

293. Let us take the general equation

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = o,$$

in which $x$ and $y$ represent rectangular co-ordinates. Let us seek the form and position of the curves which it represents, according to the different values of the independent coefficients A, B, C, D, E, F. Resolving this equation with respect to $y$, we have

$$y = -\frac{Bx + D}{2A} \pm \frac{1}{2A}\sqrt{(B^2 - 4AC)x^2 + 2(BD - 2AE)x + D^2 - 4AF}.$$

In consequence of the double sign of the radical, there will, in general, be two ordinates corresponding to the same abscissa, which we may determine and construct, if the values given to $x$, under the radical, be *real.* If this reduce it to zero, there will be but one value of $y$, and if they render it imaginary, there will be no point of the curve corresponding to these abscissas.

Hence to determine the extent of the curve in the direc-

tion of the axis of $x$, we must seek whether the values given to $x$ render the radical, *real*, *zero*, or *imaginary*.

294. In this discussion we will suppose that the general equation contains the second power of at least one of the variables $x$ or $y$. For, if the equation were independent of these terms, its discussion would be rendered very simple, and the curve which it represents immediately determined. The general equation under this supposition would reduce to

$$Bxy + Dy + Ex + F = o,$$

which may be put under the form

$$B\left(x + \frac{D}{B}\right)\left(y + \frac{E}{B}\right) - \frac{DE}{B} + F = o,$$

and making

$$x + \frac{D}{B} = x', \; y + \frac{E}{B} = y',$$

it becomes

$$x'y' = \frac{DE}{B} + F,$$

which is the equation of an hyperbola referred to its asymptotes.

295. The result would be still more simple if the coefficients A, B, C, reduced the three terms in $x^2$, $y^2$, and $xy$, to zero. In this case the general equation would become of the first degree, and would evidently represent a straight line, which could be readily constructed. These particular cases presenting no difficulty, we will suppose in this discussion that the square of the variable $y$ enters into the general equation.

296. Resuming the value of $y$ deduced from the general equation,

$$y = -\frac{Bx+D}{2A} \pm \frac{1}{2A}\sqrt{(B^2-4AC)x^2+2(BD-2AE)x+D^2-4AF},$$

we see that the circumstances which determine the reality of $y$ depend upon the sign of the quantity under the radical. But we know from Algebra, that in an expression of this kind, we can always give such a value to $x$, as to make the sign of this polynomial depend upon that of the first term: and since $x^2$ is positive for all values of $x$, the sign will depend upon that of the quantity $(B^2 - 4AC)$. We may therefore conclude,

1st. *When* $B^2 - 4AC$ *is negative*, there will be values of $x$ both positive and negative, for which the values of $y$ will be imaginary, The curve is therefore limited on both sides of the axis of $y$.

2dly. *When* $(B^2 - 4AC) = o$, the first term of the polynomial disappears, and the sign of the polynomial will then depend upon that of the second term $(BD - 2AE)\,x$. If $(BD - 2AE)$ be positive, the curve will extend indefinitely for all values of $x$ that are positive. But if $x$ be negative, $y$ becomes imaginary. The curve is therefore limited on the side of the negative abscissas. The reverse will be the case if $(BD - 2AE)$ is negative. The curve will in this case extend indefinitely when $x$ is negative, and be limited for positive values of $x$.

3dly. *When* $(B^2 - 4AC)$ *is positive*, there will be positive and negative values for $x$, beyond which those of $y$ will be always real. The curve will therefore extend indefinitely in both directions.

297. We are therefore led to divide curves of the second order into three classes, to wit,

1. Curves limited in every direction;

Character, . . . $B^2 - 4AC < o$.

2. Curves limited in one direction, and indefinite in the opposite;

Character, . . . $B^2 - 4AC = o$.

3. Curves indefinite in all directions;

Character, . . . $B^2 - 4AC > o$.

The ellipse is comprehended in the first class, the parabola in the second, and the hyperbola in the third. We will discuss each of these classes.

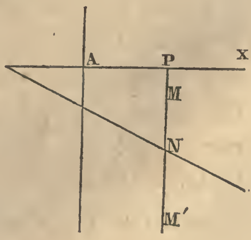
FIRST CLASS.—*Curves limited in every direction.*

*Analytical Character*, $B^2 - 4AC < o$.

298. Let us resume the general value of $y$,

$$y = -\frac{Bx + D}{2A} \pm \frac{1}{2A}\sqrt{(B^2-4AC)x^2+2(BD-2AE)x+D^2-4AF}.$$

This expression shows, that, to find points in the curve, we must construct for every abscissa AP an ordinate equal to $-\left\{\frac{Bx + D}{2A}\right\}$ which will determine a point N, above and below which we must lay off the distance represented by the radical. From which it follows that each of the points N bisects the corresponding line MM′, which is limited by the curve. This quantity $-\left\{\frac{Bx + D}{2A}\right\}$



which varies with $x$, is the ordinate of a straight line whose equation is

$$y = -\left\{\frac{Bx + D}{2A}\right\}.$$

This line is, therefore, the locus of the points N, which we have just considered. Hence, it bisects all the lines drawn parallel to the axis of $y$ and limited by the curve. This line is called the *diameter* of the curve.

299. Let us now determine the limit of the curve in the direction of the axis of $x$. For this purpose we may put the polynomial under the radical under another form,

$$y = -\frac{Bx + D}{2A} \pm$$

$$\frac{1}{2A}\sqrt{(B^2 - 4AC)\left(x^2 + 2\frac{BD - 2AE}{B^2 - 4AC}x + \frac{D^2 - 4AF}{B^2 - 4AC}\right)},$$

and if we represent by $x'$ and $x''$ the two roots of the equation

$$x^2 + 2\frac{BD - 2AE}{B^2 - 4AC}x + \frac{D^2 - 4AF}{B^2 - 4AC} = o,$$

the value of $y$ will take the form

$$y = -\frac{Bx + D}{2A} \pm \frac{1}{2A}\sqrt{(B^2 - 4AC)(x - x')(x - x'')}.$$

Hence we see, the values of $y$ will be real or imaginary according to the signs of the factors $(x - x')$ and $(x - x'')$, and consequently, the limits of the curve will depend upon the values of $x'$ and $x''$. These values may be *real and unequal*, *real and equal*, or *imaginary*. We will examine these three cases.

300. 1st. *If the roots are real and unequal*, all the values

of $x$ greater than $x'$ and less than $x''$, will give contrary signs to the factors $x - x'$, $x - x''$, and this product will be negative, but as $B^2 - 4AC$ is also negative, the quantity $(B^2 - 4AC)(x - x')(x - x'')$ will be positive, and the ordinate $y$ will have two real values. If we make $x = x'$ or $x = x''$, the radical will disappear, the two values of $y$ will be real and equal to $-\frac{Bx + D}{2A}$. In this case the abscissas $x'$ and $x''$ belong to the points in which the curve meets its diameter, that is, to the vertices of the curve. Finally, for $x$ positive or negative, but greater than $x'$ and $x''$, the two factors $(x - x')$, $(x - x'')$, will be positive, as well as their product $(x - x')(x - x'')$; and since $B^2 - 4AC$ is negative, the quantity $(B^2 - 4AC)(x - x')(x - x'')$ will be negative also, and both values of $y$ will be imaginary.

301. We see from this discussion that the curve is continuous between the abscissas $x'$, $x''$, but does not extend beyond them; and if at their extremities we draw two perpendiculars to the axis of $x$, these lines will limit the curve, and be tangent to it, since we may regard them as secants whose points of intersection have united.

302. By resolving the equation with respect to $x$, we would arrive at similar conclusions, and the limits of the curve in the direction of the axis of $y$, would be the tangents to the curve drawn parallel to the axis of $x$.

303. Having thus found four points of the curve, we could ascertain the points in which the curve cuts the co-ordinate axes. By making $x = o$, we have

$$Ay^2 + Dy + F = o,$$

and the roots of this equation will give the points in which the curve cuts the axis of $y$. According as the values of $y$

are real and unequal, real and equal, or imaginary, the curve will have two points of intersection with the axis of $y$, be tangent to it, or not meet it at all.

304. By making $y = o$, we have

$$Cx^2 + Ex + F = o,$$

and the roots of this equation will in the same manner determine the points in which the curve cuts the axis of $x$.

305. In comparing this curve with those of the Conic Sections, we see at once its similarity to the Ellipse. Its position will depend upon the particular values of the coefficients A, B, C, &c.

306. Let us apply these principles to a numerical example, and construct the curve represented by the equation

$$y^2 - 2xy + 2x^2 - 2y + 2x = o.$$

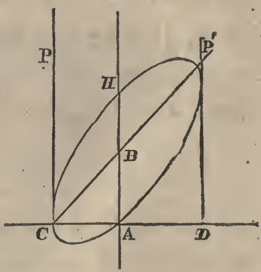
Resolving this equation with respect to $y$, we have

$$y = (x + 1) \pm \sqrt{(x + 1)^2 - 2x\,(x + 1)}.$$

The equation

$$y = (x + 1),$$

is that of the diameter of the curve, and laying off on the axis of $y$ a distance AB equal to 1, and drawing BC making an angle of 45° with the axis of $x$, BC will be this diameter. The roots of the equation




$$(x + 1)^2 - 2x\,(x + 1) = o$$

are

$$x = +1, \qquad x = -1.$$

Laying off on both sides of the axis of $y$ distance AC and AD equal to 1, the perpendiculars CP, DP′, will limit the curve in this direction. Substituting the values of $x$ in the preceding equation, we have the corresponding values of $y$,

$$y = +2, \qquad y = o.$$

The first gives the point P′, the second the point C.

Making $x = o$, the equation becomes

$$y^2 - 2y = o,$$

which gives

$$y = o, \qquad y = +2,$$

for the points A and H, in which the curve cuts the axis of $x$.

For $y = o$, we have

$$x^2 + x = o,$$

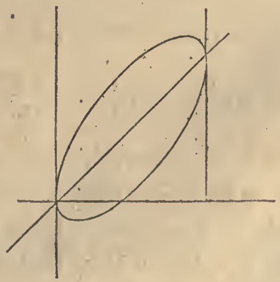
and

$$x = o, \qquad x = -1,$$

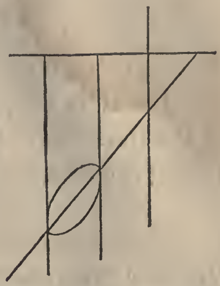
corresponding to the points A and C on the axis of $x$.

The following examples may be discussed in the same manner:

2. $y^2 - 2xy + 2x^2 - 2x = o.$



3. $y^2 - 2xy + 2x^2 + 2y + x + 3 = o.$



307. There is a particular case comprehended under this class, which it would be well to examine. It is that in which $A = C$ and $B = o$ in the general equation. This supposition gives

$$Ay^2 + Ax^2 + Dy + Ex + F = o;$$

or dividing by A,

$$y^2 + x^2 + \frac{D}{A}y + \frac{E}{A}x + \frac{F}{A} = o.$$

If we add $\frac{D^2 + E^2}{4A^2}$ to both sides of this equation, it may be put under the form

$$\left\{y + \frac{D}{2A}\right\}^2 + \left\{x + \frac{E}{2A}\right\}^2 = \frac{D^2 + E^2 - 4AF}{4A^2}.$$

If the co-ordinates $x$, $y$, are rectangular, this equation is of the same form as that in Art. and therefore represents a circle, the co-ordinates of whose centre are $-\frac{D}{2A}$, $-\frac{E}{2A}$, and whose radius is $\frac{\sqrt{D^2 + E^2 - 4AF}}{2A}$. In order that this circle be real, it is necessary that the quantity $(D^2 + E^2 - 4AF)$ be positive. If $D^2 + E^2 - 4AF = o$, the circle reduces to a point. If the system of co-ordinates be oblique, this equation will be that of an ellipse.

308. We come now to the second supposition, in which *the roots $x'$, $x''$, are equal.* The product $(x - x')(x - x'')$ becomes $(x - x')^2$, and the general value of $y$ is

$$y = -\frac{Bx + D}{2A} \pm \frac{x - x'}{2A}\sqrt{B^2 - 4AC.}$$

Whatever value we give to $x$ which does not reduce $x - x'$ to zero, will give imaginary values for $y$, since $B^2 - 4AC$ is negative. But if $x = x'$, there will be but one value for $y$, which will be real and equal to $-\left\{\frac{Bx + D}{2A}\right\}$. In this case the curve rednces to a single point, situated on the diameter, the co-ordinates of which are

$$x = x', \quad y = -\left\{\frac{Bx + D}{2A}\right\}.$$

EXAMPLES.

$$x^2 + y^2 = o, \qquad y^2 + x^2 - 2x + 1 = o.$$

309. Finally, when *the roots are imaginary.* In this case the product $(x - x')(x - x'')$ will always have the same sign, whatever value be given to $x$. But we can always take $x$ sufficiently large to render this product positive, since the first term is $+ x^2$. The product $(x - x')(x - x'')$ will therefore always be positive, and as $(B^2 - 4AC)$ is negative, it follows that the values of $y$ will always be imaginary, and there will be no curve.

EXAMPLES.

$$y^2 + xy + x^2 + \tfrac{1}{2}x + y + 1 = o, \quad y^2 + x^2 + 2x + 2 = o,$$

which may be put under the forms

$$(2y + x + 1)^2 + 3x^2 + 3 = o, \quad y^2 + (x + 1)^2 + 1 = o.$$

310. There results from the preceding discussion, that the curve of the second order, comprehended in the first class, for which $B^2 - 4AC$ is negative, are in general re-entrant curves as the ellipse, but the secondary conditions give rise to three varieties, which are *the Point*, *the Imaginary Curve*, and *the Circle*.

SECOND CLASS.—*Curves limited in one direction and indefinite in the opposite.*

*Analytical Character*, $B^2 - 4AC = o$.

311. In this case the general value of $y$ becomes

$$y = -\frac{Bx + D}{2A} \pm \frac{1}{2A}\sqrt{2(BD - 2AE)x + D^2 - 4AF}.$$

Making, for more simplicity,

$$\frac{D^2 - 4AF}{2(BD - 2AE)} = -x',$$

it may be put under the form

$$y = -\frac{Bx + D}{2A} \pm \frac{1}{2A}\sqrt{2(BD - 2AE)(x - x')}.$$

If $BD - 2AE$ is positive, so long as $x$ is greater than $x'$, the factor $x - x'$ will be positive, and the radical will be real. If $x = x'$, the radical will disappear, and if $x$ be less than $x'$, the factor $x - x'$ will be negative, and the radical will be imaginary. The curve therefore extends indefinitely

from $x = x'$ to $x = +$ infinity. The ordinate corresponding to $x = x'$, will be tangent to the curve at this point.

312. The contrary will be the case if BD — 2AE is negative. The curve will extend indefinitely on the side of the negative abscissas, and will be limited in the opposite direction.

In both cases, the straight line whose equation is

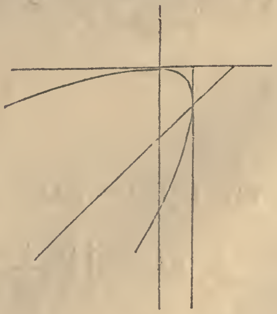
$$y = -\frac{Bx + D}{2A}$$

will be the *diameter* of the curve.

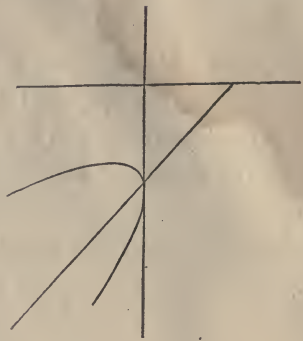
EXAMPLES.

1. $y^2 - 2xy + x^2 + x = o.$

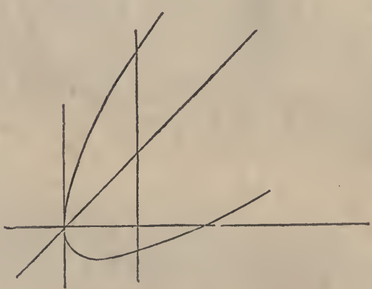
2. $y^2 - 2xy + x^2 + 2y = o.$



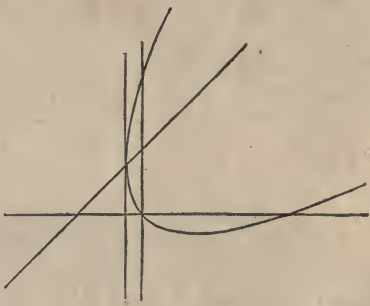
3. $y^2 - 2xy + x^2 + 2y + 1 = o.$



4. $y^2 - 2xy + x^2 - 2y - 1 = o.$



5. $y^2 - 2xy + x^2 - 2y - 2x = o.$



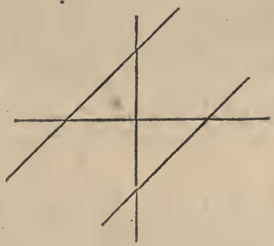
313. If $BD - 2AE = o$, the value of $y$ becomes

$$y = -\left\{\frac{Bx + D}{2A}\right\} \pm \frac{1}{2A}\sqrt{D^2 - 4AF}.$$

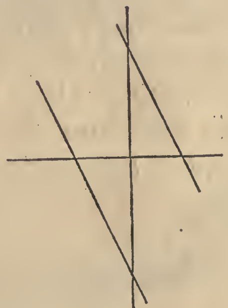
The curve becomes two parallel straight lines, which will be *real, one straight line*, or *two imaginary lines*, according as $D^2 - 4AF$ is *positive, nothing*, or *negative*.

EXAMPLES.

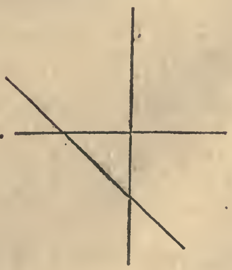
1. $y^2 - 2xy + x^2 - 1 = o$.



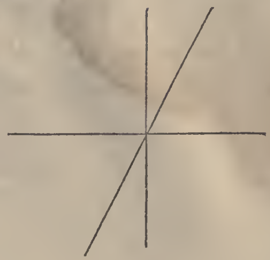
2. $y^2 + 4xy + 4x^2 - 4 = o$.



3. $y^2 - 2xy + x^2 + 2y - 2x + 1 = o$.

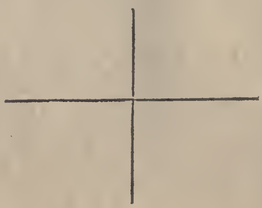


4. $y^2 - 4xy + 4x^2 = o.$



5. $y^2 + 2xy + x^2 + 1 = o.$

6. $y^2 + y + 1 = o.$



314. There results from this discussion, that the curves of the second order, comprehended in the second class, for which $B^2 - 4AC = o$, are in general indefinite in one direction as the parabola, but include as varieties *two parallel straight lines*, *one straight line*, and *two imaginary straight lines.*

THIRD CLASS.—*Curves indefinite in every direction.*

*Analytical Character*, $B^2 - 4AC > o.$

315. The discussion of this class of curves presents no difficulty, as the method is precisely similar to that of the first class. Resuming the general value of $y$,

$$y = -\frac{Bx + D}{2A} \pm \frac{1}{2A}\sqrt{x^2 + 2\,\frac{BD - 2AE}{B^2 - 4AC}\,x + \frac{D^2 - 4AF}{B^2 - 4AC}},$$

and representing by $x'$, $x''$, the roots of the equation

$$x^2 + 2\frac{BD - 2AE}{B^2 - 4AC} + \frac{D^2 - 4AF}{B^2 - 4AC} = o,$$

the value of $y$ becomes

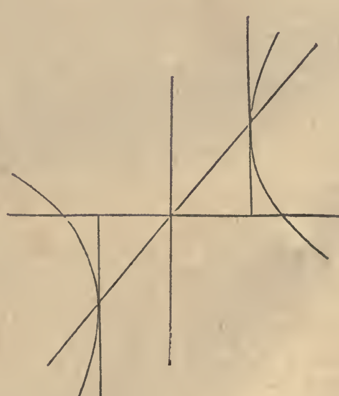
$$y = -\frac{Bx + D}{2A} \pm \frac{1}{2A}\sqrt{(B^2 - 4AC)(x - x')(x - x'')}.$$

So long as $x'$ and $x''$ are real, the curve will be imaginary between the limits $x'$, $x''$, since $(B^2 - 4AC)$ is positive, but for all values of $x$, positive as well as negative, beyond this limit, the values of $y$ will be real. The abscissas $x'$, $x''$, correspond to the points in which the curve intersects its diameter; and the equation of this diameter is,

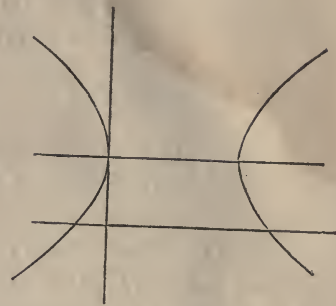
$$y = -\frac{Bx + D}{2A}.$$

EXAMPLES.

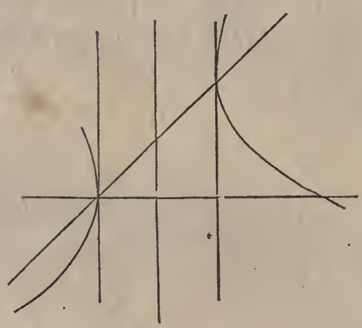
1. $y^2 - 2xy - x^2 + 2 = o$.



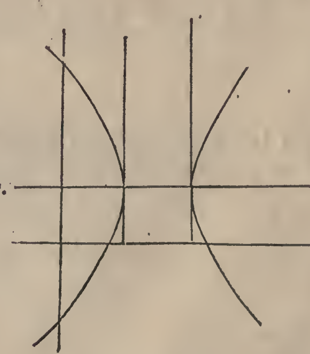
2. $y^2 - x^2 - 2x - 2y + 1 = o.$



3. $y^2 - 2xy - x^2 - 2y +$
$2x + 3 = o.$



4. $y^2 - 2x^2 - 2y + 6x - 3 = o.$



316. We may find the points in which the curve cuts the axes by the methods pursued in Arts. 304 and 305.

317. When the roots $x'$, $x''$, are equal, the product $(x - x')(x - x'')$ would reduce to $(x - x')^2$, and we would have

$$y = -\frac{Bx + D}{2A} \pm \frac{x - x'}{2A}\sqrt{B^2 - 4AC}.$$

This equation represents two straight lines, which are always real, since $B^2 - 4AC$ is positive.

EXAMPLES.

1. $y^2 - 2x^2 + 2y + 1 = o.$

2. $y^2 - x^2 = o.$

3. $y^2 + xy - 2x^2 + 3x - 1 = o.$

318. When $x'$ and $x''$ are imaginary, the quantity under the radical will be always positive, since a value may be given to $x$ to make $(x - x')(x - x'')$ always positive, and $B^2 - 4AC$ is positive for this class of curves. Hence, whatever value we give to $x$, that of $y$ will be real, and will give points of the curve. This curve will be composed of two separate branches, and the line represented by the equation

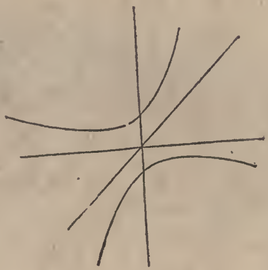
$$y = -\frac{Bx + D}{2A}$$

will be its diameter.

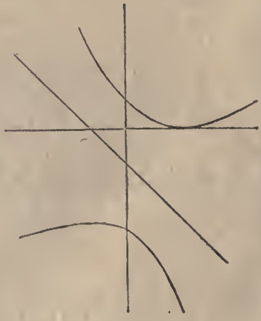
As the radical $\sqrt{(B^2 - 4AC)(x - x')(x - x'')}$ can never reduce to zero, this diameter does not cut the curve.

EXAMPLES.

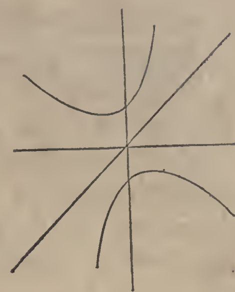
1. $y^2 - 2xy - x^2 - 2 = o.$



2. $y^2 + 2xy - x^2 + 2x + 2y - 1 = o.$



3. $y^2 - 2xy - x^2 - 2x - 2 = o.$



319. If $A = -C$, and $B = o$, the general equation becomes

$$Ay^2 - Ax^2 + Dy + Ex + F = o,$$

or,

$$y^2 - x^2 + \frac{D}{A}y + \frac{E}{A}x + \frac{F}{A} = o,$$

which may be put under the form

$$\left(y + \frac{D}{2A}\right)^2 - \left(x - \frac{E}{2A}\right)^2 = \frac{D^2 - E^2 - 4AF}{4A^2}.$$

Hence we see, that if the co-ordinates $x$ and $y$ are rectangular, this equation represents an equilateral hyperbola, the co-ordinates of whose centre are $-\frac{D}{2A}$, $+\frac{E}{2A}$, and whose power is $\frac{D^2 - E^2 - 4AF}{4A^2}$. This case is analogous to that of the circle (Art. 307).

320. We conclude from this discussion that the curves of the second order, comprehended in the third class, for which $B^2 - 4AC$ is positive, are always curves composed of two separate and infinite branches, as the hyperbola, and that they include, as varieties, *two straight lines* and the *equilateral hyperbola.*

## *Of the Centres and Diameters of Plane Curves.*

321. The *centre* of a curve is that point through which, if any line be drawn terminated in the curve, the points of intersection will be equal in number, and the line will be bisected at the centre.

322. If we suppose this condition satisfied, and that the origin of co-ordinates is transferred to this point, then it follows, that if $+x'$, $+y'$, represent the co-ordinates of one of the points in which the line drawn through the centre intersects the curve, the curve will have another point, of which the co-ordinates will be $-x'$, $-y'$, that is, its equation will be satisfied when $-x'$, $-y'$, are substituted for $+x'$, $+y'$. This condition will evidently be fulfilled if the equation of the curve contain only the even powers of the variables $x$ and $y$, for these terms will undergo no change when $-x'$ is substituted for $+x'$, and $-y'$ for $+y'$. To determine, therefore, whether a given curve has a centre, we must examine if it have a point in its plane, to which, if the curve be referred as the origin of co-ordinates, the transformed equation will contain variable terms of an even dimension only.

323. For example, to determine whether curves of the second order represented by the general equation

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = o,$$

have centres, we must substitute for $x$ and $y$, expressions of the form

$$x = a + x', \quad y = b + x',$$

in which $a$ and $b$ are the co-ordinates of the new origin, and

see whether we can dispose of these quantities in such a manner as to cause every term of an uneven dimension to disappear from the transformed equation. If this substitution be made, the transformed equation will generally contain two terms of an uneven dimension, to wit, $(2\mathrm{A}b + \mathrm{B}a + \mathrm{D})\,y'$ and $(2\mathrm{C}a + \mathrm{B}b + \mathrm{E})\,x'$. And in order that these terms disappear, $a$ and $b$ must be susceptible of such values as to make

$$2\mathrm{A}b + \mathrm{B}a + \mathrm{D} = o, \quad 2\mathrm{C}a + \mathrm{B}b + \mathrm{E} = o,$$

and then the equation referred to the new origin becomes

$$\mathrm{A}y'^2 + \mathrm{B}x'y' + \mathrm{C}x'^2 + \mathrm{A}b^2 + \mathrm{B}ab + \mathrm{C}a^2 + \mathrm{D}b + \mathrm{E}a + \mathrm{F} = o;$$

and under this form we see that it undergoes no change when $-x'$, $-y'$, are substituted for $+x'$, $+y'$.

324. The relations which exist between the co-ordinates $a$ and $b$ are of the first degree, and represent two straight lines. These lines can only intersect in one point. *Hence, curves of the second order have only one centre.*

325. In fact these equations give for $a$ and $b$, the following values,

$$a = \frac{2\mathrm{AE} - \mathrm{BD}}{\mathrm{B}^2 - 4\mathrm{AC}}, \quad b = \frac{2\mathrm{CD} - \mathrm{BE}}{\mathrm{B}^2 - 4\mathrm{AC}},$$

and these values are single. They become infinite when $\mathrm{B}^2 - 4\mathrm{AC} = o$, which shows that there is no centre, or that it is at an infinite distance from the origin, which is the case with curves of the second class. Here the two lines whose intersection determines the centre become parallel. If one of the numerators be zero at the same time with the denominator, the values of $a$ and $b$ become indeterminate. This arises from the fact, that this supposition reduces the two equations to a single one, which is not sufficient to deter-

mine two unknown quantities. There are therefore an infinite number of centres situated on the same straight line. But in this case the curve reduces to two parallel lines, and the centres are found on a line between the two.

326. The *diameter* of a curve is any straight lines which bisects all the parallel chords drawn in the curve. If, therefore, we take a diameter for the axis of $x$, and take the axis of $y$ parallel to the chords which are bisected by this diameter, the transformed equation will be such, that if it be satisfied by the values $+x'$, $+y'$, it must also be by $+x'$, $-y'$, that is, by the same ordinate taken in an opposite direction. Consequently, to ascertain whether a curve has one or more diameters, we must change the diameters of the axes by means of the general formulas

$$x = a + x' \cos\alpha + y' \cos\alpha', \;\; y = b + x' \sin\alpha + y' \sin\alpha',$$

and after substituting these values we must determine $a$, $b$, $\alpha$, $\alpha'$, in such a manner, that all the terms affected with uneven powers of one of the variables disappear, without the variables themselves ceasing to be indeterminate. If this be possible, the direction of the other variables will be a diameter of the curve.

327. Let us apply these principles to the general equation

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = o.$$

Making the substitutions, we shall find, that the transformed equation will generally contain three terms, in which one of the variables $x'$, $y'$, will be of an uneven degree, and these terms are

$$\{2A \sin\alpha \sin\alpha' + B(\sin\alpha \cos\alpha' + \sin\alpha' \cos\alpha) + 2C \cos\alpha \cos\alpha'\}\, x'y'$$

$$+ \{(2Ab + Ba + D)\sin\alpha + (2Ca + Bb + E)\cos\alpha\}\, x'$$
$$+ \{(2Ab + Ba + D)\sin\alpha' + (2Ca + Bb + E)\cos\alpha'\}\, y'.$$

Now, if we wish to render $x'$ a diameter, the co-efficients of the terms in $y'$ must disappear, which requires that we make

$$\{2A\sin\alpha\sin\alpha' + B(\sin\alpha\cos\alpha' + \sin\alpha'\cos\alpha) + 2C\cos\alpha \cos\alpha'\}\, x'y' = o\,;$$

or, what is the same thing,

$$2C + B(\text{tang}\,\alpha' + \text{tang}\,\alpha) + 2A\,\text{tang}\,\alpha\,\text{tang}\,\alpha' = o, \quad (1).$$

and that we also have

$$\{(2Ab + Ba + D)\sin\alpha' + (2Ca + Bb + E)\cos\alpha'\}\, y' = o. \ (2).$$

If, on the contrary, we wished the axis of $y'$ to be a diameter, the co-efficients of the terms in $x'$ must disappear. But this supposition would also require equation (1) to be satisfied, and that, in addition to this, we have

$$\{(2Ab + Ba + D)\sin\alpha + (2Ca + Bb + E)\cos\alpha\}\, x' = o. \ (3).$$

328. Let us examine what these equations indicate.

We see in the first place, that whichever axis we select for a diameter, equation (1) must always exist, and it is also necessary to connect with it one of the equations (2) or (3). The first equation determines the relation between $\alpha$ and $\alpha'$, and when one of them is given, it assigns a real value to the other. But after this equation is thus satisfied, the second equation (2) or (3) which is connected with it, can only be fulfilled by giving proper values to $a$ and $b$; so that while equation (1) assigns a direction to the chords which are bisected by the diameter, equation (2) or (3) between $a$ and $b$, will be the equation of this diameter relatively to the first co-ordinate axes.

329. Equations (2) and (3) are evidently both satisfied when we make

$$2Ab + Ba + D = o, \qquad 2Ca + Bb + E = o. \quad (4).$$

Hence the values of $a$ and $b$ given by these conditions belong to a point which is common to every diameter. But these conditions are the same as those which determine the centre (Art. ).

Hence every diameter of curves of the second order passes through the centre, and reciprocally every line drawn through the centre is a diameter.

330. If both of the axes $x'$, $y'$, be diameters, the transformed equation will not contain the uneven powers of either of the variables. For equations, (1), (2), and (3) must in this case exist.

331. This condition is always fulfilled in curves of the second order, when the origin of the co-ordinate axes is taken at the centre, and their direction satisfies equation (1). For, in this case, the first powers of $x'$ and $y'$ having disappeared, as well as the term in $x'y'$, the equation will contain only the square powers of the variables. These systems of diameters are called *Conjugate Diameters*. But the condition of passing through the centre really limits this property to the Ellipse and Hyperbola, the only cases in which equation (4) can be satisfied for finite values of $a$ and $b$.

332. When the transformed equation contains only even powers of the variables, it is evident that if this equation be satisfied by the values $+ x'$, $+ y'$, it will also be for $- x'$, $+ y'$; $- x'$, $- y'$; $+ x'$, $- y'$; that is, in the four angles of the co-ordinate axes, there will be a point whose co-ordinates will only vary in signs. If the axes be rectangular, the form of the curve will be identically the same in each of

these angles. In this case, it is said to be *symmetrical* with respect to the axes. In the ellipse and hyperbola, for example, these curves are symmetrically situated, when the co-ordinate axes coincide with the axes of the curves. When $x'$ and $y'$ are at right angles, we have $\alpha' = \alpha + 90^\circ$, and eliminating $\alpha'$ from equation (1), we have

$$-2C \sin\alpha \cos\alpha + B(\cos^2\alpha - \sin^2\alpha) + 2A \sin\alpha \cos\alpha = o,$$

and

$$(A - C) \text{ tang } {}^2\alpha + B = o,$$

an equation which will always give a real value for tang $^2\alpha$, from which we deduce two real values for tang $\alpha$. But these two values will be such, that if one be $\alpha$, the other will be $90^\circ + \alpha$, consequently the system of co-ordinate axes will be the two axes.

### *Identity of Curves of the Second Degree, with the Conic Sections.*

333. The curves which have been discovered in the discussion of the general equation of the second degree, have presented a striking analogy to the Conic Sections. We will resume this equation, and see how far this analogy extends.

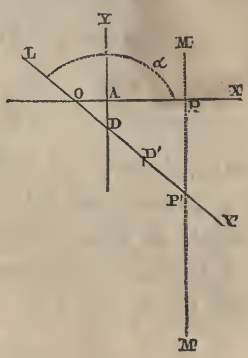
334. We will suppose the equation to contain the second power of at least one of the variables, and that the system of axes is rectangular. We have found for the general value of $y$ (Art. ),

$$y = -\frac{1}{2A}(Bx+D) \pm \frac{1}{2A}\sqrt{(B^2-4AC)x^2+2(BD-2AE)+D^2-4AF.}$$

The expression

$$y = -\frac{1}{2A}(Bx + D),$$

is the equation of the diameter of the curve, and the radical expresses the ordinate of the curve counted from this diameter. Let us construct these results. The diameter cuts the axis of $y$ at a distance from the origin equal to $-\frac{D}{2A}$, and makes an angle with the axis of $x$, the trigonometrical tangent of which is $-\frac{B}{2A}$. Laying off a length $AD = -\frac{D}{2A}$, and through D drawing LDX′, making the angle LOX equal to that whose tangent is $-\frac{B}{2A}$, LDX′ will be the diameter of the curve.




Let us now consider any point M whose abscissa AP $= x$, and ordinate PM $= y$. Produce PM until it meets the diameter OX′, the distance PP′ will represent $-\frac{1}{2A}(Bx+D)$ and PM the radical part of the value of $y$. But as the equation of a curve is simplified by referring it to its diameter, let us refer the curve to new co-ordinates, of which DP′ $= x'$ and P′M $= y'$, and call the angle LOX, $\alpha$, we have

$$x = -x' \cos\alpha, \quad y = -\frac{1}{2A}(Bx + D) + y',$$

Substituting these expressions in the general value of $y$, we get

$$y' = \frac{1}{2A}\sqrt{(B^2-4AC)\cos^2\alpha x'^2-2(BD-2AE)\cos\alpha x'+D^2-4AF},$$

or, squaring both members,

$$4A^2y'^2 = (B^2 - 4AC)\cos^2\alpha \,.\, x'^2 - 2(BD - 2AE)$$
$$\cos\alpha \,.\, x' + D^2 - 4AF, \qquad (2.)$$

or,

$$4A^2y'^2 = (B^2-4AC)\cos^2\alpha \left\{ x'^2 - 2\frac{(BD - 2AE)\,x'}{(B^2 - 4AC)\cos\alpha} \right\} + D^2 - 4AF.$$

Adding $\frac{(BD - 2AE)^2}{(B^2 - 4AC)^2\cos^2\alpha}$ to the quantity within the parentheses, and subtracting without the parentheses the equivalent $(B^2 - 4AC)\cos^2\alpha\,\frac{(BD - 2AE)^2}{(B^2 - 4AC)^2\cos^2\alpha}$, the equation becomes

$$4A^2y'^2 = (B^2 - 4AC)\cos^2\alpha \left\{ x' - \frac{BD - 2AE}{(B^2 - 4AC)\cos\alpha} \right\}^2 -$$
$$\frac{(BD - 2AE)^2}{B^2 - 4AC} + D^2 - 4AF.$$

Let us introduce for $x'$ a new variable $x''$, such that

$$x' - \frac{BD - 2AE}{(B^2 - 4AC)\cos\alpha} = x'',$$

which is the same thing as transferring the origin of co-ordinates from the point D to D', so that $DD' = \frac{BD - 2AE}{(B^2 - 4AC)\cos\alpha}$. The equation in $y'$ and $x''$ becomes

$$4A^2y'^2 = (B^2-4AC)\cos^2\alpha . x''^2 - \frac{(BD-2AE)^3}{B^2-4AC} + D^2 - 4AF. \quad (3.)$$

And since under this form it contains only the squares, pow-

ers of the variables, and a constant term, we see that it can only represent an ellipse or hyperbola, referred to their centre and axes, or conjugate diameters. It will represent an ellipse if $B^2 - 4AC$ is negative, and the hyperbola if it is positive.

335. This reduction supposes that the last transformation is possible. But this will always be the case, unless $\frac{BD - 2AE}{B^2 - 4AC) \cos \alpha}$, which represents $DD'$, become *infinite*, which can only be the case when $(B^2 - 4AC) \cos \alpha = o$. But $\cos \alpha$ cannot be zero, for then we should have $\alpha = 90°$, which would make $A = o$, and the diameter $DX'$ parallel to the primitive axis of $y$, a case which we excluded at first; hence, in order that $DD'$ = infinity, we must have $B^2 - 4AC = o$, and this reduces the transformed equation to

$$4A^2y'^2 = -2(BD - 2AE) \cos \alpha . x' + D^2 - 4AF, \quad (4.)$$

which is the equation of a parabola referred to its diameter $DX'$. Thus, in every possible case, the equation of the second degree between two indeterminates can only represent one or the other of the conic sections.

336. All the particular cases which the conic sections present may be deduced from these transformations. For example, if in equation (4) we suppose $BD - 2AE = o$, the term in $x'$ disappears, and the parabola is changed into two straight lines parallel to the axis of $x'$. If $D^2 - 4AF = o$ also, the equation will represent but one straight line, which coincides with this axis. If in equation (3), we make different suppositions upon the quantities A, B, C, D, and E, we may deduce all the known varieties of the sections which this equation represents, which proves the perfect identity of every curve of the second order with the conic sections.

# CHAPTER VII.

## OF SURFACES OF THE SECOND ORDER.

337. Surfaces, like lines, are divided into orders, according to the degree of their equations. The plane, whose equation is of the first degree, is a surface of the *first order*.

338. We will here consider surfaces of the *second order*, the most general form of its equation being

$$Az^2 + A'y^2 + A''x^2 + Byz + B'xz + B''xy + Cz +$$
$$C'y + C''x + F = o. \qquad (1.)$$

Since two of the variables $x$, $y$, $z$, may be assumed at pleasure, if we find the value of one of them, as $z$, in terms of the other two, we could, by giving different values to $x$ and $y$, deduce the corresponding values of $z$, and thus determine the position of the different points of the surface. But as this method of discussion does not present a good idea of the form of the surfaces, we shall make use of another method, which consists in intersecting the surface by a series of planes, having given positions with respect to the co-ordinate axes. Combining then the equations of these planes with that of the surface, we determine the curves of intersections, whose position and form will make known the character of the given surface.

339. To exemplify this method, take the equation

$$x^2 + y^2 + z^2 = R^2,$$

and let this surface be intersected by a plane, parallel to the plane of $xy$; its equation will be of the form (Art. 76),

$$x = a,$$

and substituting this value of $x$, in the proposed equation, we have

$$x^2 + y^2 = R^2 - a^2,$$

for the equation of the projection of the intersection of the plane and surface on the plane of $xy$. It represents a circle (Art. 148), whose centre is at the origin, and whose radius is $\sqrt{R^2 - a^2}$. This radius will be *real*, *zero*, or *imaginary*, according as $a$ is less than, equal to, and greater than, R. In the first case the intersection will be the circumference of a circle, in the second the circle is reduced to a point, and in the third the plane does not meet the surface.

340. The proposed equation being symmetrical with respect to the variables $x$, $y$, $z$, we shall obtain similar results by intersecting the surface by planes parallel to the other co-ordinate planes. It is evident, then, that the surface is that of a sphere.

341. The co-ordinate planes intersect this surface in three equal circles, whose equations are,

$$x^2 + y^2 = R^2,\ x^2 + z^2 = R^2,\ y^2 + z^2 = R^2.$$

342. We may readily see that the expression $\sqrt{x^2 + y^2 + z^2}$ represents a spirical surface, since it is the distance of any point in space from the origin of co-ordinates (Art. 80), and as this distance is constant, the points to which it corresponds are evidently on the surface of a spire, having its centre at the origin of co-ordinates.

343. The discussion has been rendered much more simple, by taking the cutting planes, parallel to the co-ordinate planes, since the projections of the intersections do not differ

from the intersections themselves. Had these planes been subjected to the single condition of passing through the origin of co-ordinates, the form of their equations would have been

$$Ax + By + Cz = o;$$

and combining this with the proposed equation, we should have,

$$(A^2 + C^2)\,x^2 + 2ABxy + (B^2 + C^2)\,y^2 = R^2C^2,$$

which is the equation of the projection of the intersection on the plane of $xy$. This projection is an ellipse, but we can readily ascertain that the intersection itself is the circumference of a circle, by referring it to co-ordinates taken in the cutting plane.

**344.** We may in the same manner determine the character of any surface, by intersecting it by a series of planes, and it is evident that these intersections will, in general, be of the same order as the surface, since their equations will be of the second degree.

**345.** Before proceeding to the discussion of the general equation

$$Az^2 + A'y^2 + A''x^2 + Byz + B'xy + B''xy + Cz + \\ C'y + C''x + F = o,$$

let us simplify its form, by changing the origin, so that we have, between the two systems of co-ordinates, the relations (Art. 127),

$$x = x' + \alpha,\ y = y' + \beta, \quad z = z' + \gamma.$$

As $\alpha$, $\beta$, $\gamma$, are indeterminate, we may give such values to

them as to cause the terms of the transformed equation affected with the first power of the variables to disappear. This requires that we have

$$\begin{aligned} 2A\gamma + B\beta + B'\alpha + C &= o, \\ 2A'\beta + B''\alpha + B\gamma + C' &= o, \\ 2A''\alpha + B'\gamma + B''\beta + C'' &= o; \quad (2.) \end{aligned}$$

and, representing all the known terms in the transformed equation by L, it becomes

$$Az'^2 + A'y'^2 + A''x'^2 + Bz'y' + B'z'x' + B''x'y' + L = o. \quad (3.)$$

As all the terms in this equation are of an even degree, its form will not be changed, if we substitute $-x'$, $-y'$, $-z'$, for $+x'$, $+y'$, $+z'$. If, then, a line be drawn through the origin of co-ordinates, the points in which it meets the surface will have equal co-ordinates with contrary signs. This line is therefore bisected at the origin, which will be the *centre* of the surface, if we attribute the same signification to this point in reference to surfaces that we have for curves.

346. The equations (2) which determine the position of the centre being *linear*, they will always give real values for $\alpha$, $\beta$, $\gamma$; but the coefficients A, B, C, &c., may have such relations as to make these values *infinite*. In this case the centre of the surface will be at an infinite distance from the origin, which will take place when

$$AB''^2 + A'B'^2 + A''B^2 - BB'B'' - 4AA'A'' = o, \quad (D.)$$

which is the denominator of the values of $\alpha$, $\beta$, $\gamma$, drawn from equation (2) placed equal to zero.

347. If this condition be satisfied, and we have at the same time

$$C = o, \quad C' = o, \quad C'' = o,$$

the values of $\alpha$, $\beta$, $\gamma$, will no longer be infinite, but will become $\frac{o}{o}$, which shows that there will be an infinite number of centres. In this case the surface is a right cylinder, with an elliptic or hyperbolic base, whose axis is the *locus* of all the centres.

348. If condition (D) be not satisfied, but we have simply

$$C = o, \quad C' = o, \quad C'' = o,$$

the values of $\alpha$, $\beta$, $\gamma$, become zero, and the centre of the surface coincides with the origin. This is evident from the fact that equations (2) represent three planes, whose intersection determines the centre; and these planes pass through the origin when C, C′, C″, are zero.

349. We may still further simplify the equation (2) by referring the surface to another system of rectangular co-ordinates, the origin remaining the same, so that its equation shall not contain the product of the variables. The formulas for transformation are

$$x' = x'' \cos X + y'' \cos X' + z'' \cos X'',$$
$$y' = x'' \cos Y + y'' \cos Y' + z'' \cos Y'',$$
$$z' = x'' \cos Z + y'' \cos Z' + z'' \cos Z'',$$

with which we must add (Arts. 129 and 130),

$$\cos^2 X + \cos^2 Y + \cos^2 Z = o,$$
$$\cos^2 X' + \cos^2 Y' + \cos^2 Z' = o,$$
$$\cos^2 X'' + \cos^2 Y'' + \cos^2 Z'' = o, \quad \text{(A.)}$$

$$\cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z' = o,$$
$$\cos X \cos X'' + \cos Y \cos Y'' + \cos Z \cos Z'' = o,$$
$$\cos X' \cos X'' + \cos Y' \cos Y'' + \cos Z' \cos Z'' = o. \quad \text{(B.)}$$

Equations (B) are necessary to make the new axes rectangular. These substitutions give for the surface an equation of the form

$$Mz''^2 + M'y''^2 + M''x''^2 + Nz''y'' + N'z''x'' + N''x''y'' + P = o.$$

In order that the terms in $z''y''$, $z''x''$, $x''y''$, disappear, we must have

$$N = o, \quad N' = o, \quad N'' = o.$$

Without going through the entire operation, we can readily form the values of N, N′, N″, and putting them equal to zero, we have the following equations:

$$\left.\begin{array}{l} 2A \cos Z \cos Z' + B (\cos Z \cos Y' + \cos Y \cos Z') \\ + 2A' \cos Y \cos Y' + B' (\cos Z \cos X' + \cos X \cos Z') \\ + 2A'' \cos X \cos X' + B'' (\cos Y \cos X' + \cos X \cos Y') \end{array}\right\} = o.$$

$$\left.\begin{array}{l} 2A \cos Z \cos Z'' + B (\cos Z \cos Y'' + \cos Y \cos Z'') \\ + 2A' \cos Y \cos Y'' + B' (\cos Z \cos X'' + \cos X \cos Z'') \\ + 2A'' \cos X \cos X'' + B'' (\cos Y \cos X'' + \cos X \cos Y'') \end{array}\right\} = o. \text{(C)}$$

$$\left.\begin{array}{l} 2A \cos Z' \cos Z'' + B (\cos Z' \cos Y'' + \cos Y' \cos Z'') \\ + 2A' \cos Y' \cos Y'' + B' (\cos Z' \cos X'' + \cos X' \cos Z'') \\ + 2A'' \cos X' \cos X'' + B'' (\cos Y' \cos X'' + \cos X' \cos Y'') \end{array}\right\} = o.$$

The nine equations (A) (B) (C) are sufficient to determine the nine angles which the new axes must make with the old, in order that the transformed equation may be independent of the terms which contain the product of the varia-

bles. Introducing these conditions, the equation of the surface becomes

$$Mz''^2 + M'y''^2 + M''x''^2 + L = o, \qquad (4.)$$

which is the simplest *form for the equations of Surfaces of the Second Order which have a centre.*

350. We may express under a very simple formula surfaces with, and those without, a centre. For, if in the general equation, we change the direction of the axes without moving the origin, the axes also remaining rectangular, we may dispose of the indeterminates in such a manner as to cause the product of the variables to disappear. By this operation the proposed equation will take the form

$$Mz'^2 + M'y'^2 + M''x'^2 + Kz' + K'y' + K''x' + F = o.$$

If now we change the origin of co-ordinates without altering the direction of the axes, which may be done by making

$$z' = z'' + a, \;\; y' = y'' + a', \;\; z' = z'' + a'',$$

we may dispose of the quantities $a$, $a'$, $a''$, in such a manner as to cause all the known terms in the transformed equation to disappear. This condition will be fulfilled if the new origin be taken on the surface, and we have

$$Ma^2 + M'a'^2 + M''a''^2 + Ka + K'a' + K''a'' + F = o. \; (5.)$$

Suppressing the accents, and making, for more simplicity,

$$2Ma + K = H, \; 2M'a' + K' = H', \; 2M''a'' + K'' = H'',$$

every surface of the second order will be comprehended in the equation

$$Mz^2 + M'y^2 + M''x^2 + Hz + H'y + H''x = o. \quad (6.)$$

351. In order that equation (6) may represent surfaces which have a centre, it is necessary that the values of $a$, $a'$, $a''$, reduce this equation to the form of equation (4), which requires that the terms containing the first power of the variables disappear. This condition will always be satisfied, if the equations

$$2Ma + K = o, 2M'a' + K' = o, 2M''a'' + K'' = o,$$

give finite values for $a$, $a'$, $a''$. These values are

$$a = -\frac{K}{2M}, \; a' = -\frac{K'}{2M'}, \; a'' = -\frac{K''}{2M''},$$

and will always be finite, so long as M, M$'$, M$''$, are not zero. But if one of them, as M, be zero, the value of $a$ becomes infinite, and the surface has no centre, or this centre is at an infinite distance from the origin.

### *Of Surfaces which have a Centre.*

352. We have seen (Art. 349), that all surfaces of the second order which have a centre are comprehended in the equation

$$Mz''^2 + M'y''^2 + M''x''^2 + L = o.$$

Suppressing the accents of the variables, we have

$$Mz^2 + M'y^2 + M''x^2 + L = o.$$

Let us now discuss this equation, and examine more particularly the different kinds of surfaces which it represents.

Resolving this equation with respect to either of the vari-

ables, we shall obtain for it two equal values with contrary signs. These surfaces are therefore divided by the co-ordinate planes into two equal and symmetrical parts. The curves in which these planes intersect the surfaces are called *Principal Sections*, and the axes to which they are referred, *Principal Axes.*

If now the surface be intersected by a series of planes parallel to the co-ordinate planes, the intersections will be curves of the second order referred to their centre and axes, and the form and extent of these intersections will determine the character of the surface itself. But these intersections will evidently depend upon the signs of the co-efficients M, M$'$, M$''$, and supposing M positive, which we may always do, we may distinguish the following cases:

1st case, M$'$ and M$''$ positive,
2nd " M$'$ positive, M$''$ negative,
3d " M$'$ negative, M$''$ positive,
4th " M$'$ and M$''$ negative.

The three last cases always give two co-efficients of the same sign; they are therefore included in each other, and will lead to the same results by changing the variables in the different terms. It will be only necessary therefore to consider the first and last cases.

### CASE I.—M, M$'$, M$''$, *being positive.*

353. Let us resume the equation

$$Mz^2 + M'y^2 + M''x^2 + L = o.$$

Let this surface be intersected by planes parallel to the co-ordinate planes, their equations will be (Art. ),

$$x = \alpha,\ y = \beta,\ z = \gamma.$$

Combining these with the equation of the surface, we have

$$\begin{aligned} Mz^2 + M'y^2 + M''\alpha^2 + L &= o, \\ Mz^2 + M''x^2 + M'\beta^2 + L &= o, \\ M'y^2 + M''x^2 + M\gamma^2 + L &= o, \end{aligned}$$

for the equations of the curves of intersection. Comparing them with the form of the equation of the ellipse, we see that they represent ellipses whose centres are on the axes of $x$, $y$, and $z$.

354. To determine the *principal sections*, make

$$\alpha = o, \qquad \beta = o, \qquad \gamma = o,$$

and their equations are

$$\begin{aligned} Mz^2 + M'y^2 + L &= o, \\ Mz^2 + M''x^2 + L &= o, \\ M'y^2 + M''x^2 + L &= o, \end{aligned}$$

which also represent ellipses.

355. If $L = o$, all the sections as well as the surface reduce to a point.

If L *be positive*, the sections become imaginary, since their equation cannot be satisfied for any real values of the variables. The surface is therefore imaginary.

Finally, *if* L *be negative*, and equal to $-L'$, the sections will be real so long as

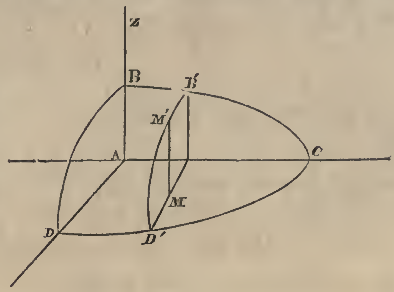
$$-L' + M''\alpha^2,\ -L' + M'\beta^2,\ -L' + M\gamma^2,$$

are negative; when these values are zero, the sections and

surface reduce to a point, and become imaginary for all values beyond this limit.

This surface is called an *Ellipsoid.*

356. If we make $y = o$ and $z = o$ in the equation of the ellipsoid, the value of $x$ will represent the abscissa of the points in which the axis of $x$ meets the surface. We find




$$x = \text{AC} = \pm \sqrt{\frac{-\text{L}}{\text{M}''}}.$$

The double sign shows that there are two points of intersections, symmetrically situated and at equal distances from the origin.

Making in the same manner $y = o$ and $x = o$, and afterwards $x = o$ and $z = o$, we obtain

$$z = \text{AB} = \pm \sqrt{\frac{-\text{L}}{\text{M}}}, \quad y = \text{AD} = \pm \sqrt{\frac{-\text{L}}{\text{M}'}}.$$

The double of these values are the *axes* of the surface, and we see that they can only be *real* when L is negative.

357. The equation of the ellipsoid takes a very simple form when we introduce the axes. Representing the semi-axes by A, B, C, we have

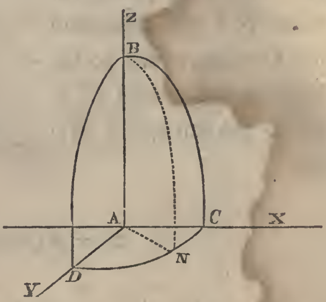
$$\text{A}^2 = -\frac{\text{L}}{\text{M}''}, \quad \text{B}^2 = -\frac{\text{L}}{\text{M}'}, \quad \text{C}^2 = -\frac{\text{L}}{\text{M}};$$

and substituting the values of M, M', M'', drawn from these equations in that of the surface, it becomes

$$A^2B^2z^2 + A^2C^2y^2 + B^2C^2x^2 = A^2B^2C^2.$$

358. If we make the cutting planes pass through the axis of $z$, and perpendicular to the plane of $xy$, their equation will be

$$y = ax,$$




or, adopting as co-ordinates the angle NAC $= \varphi$, and the radius AN$=r$, we shall have

$$x = r \cos\varphi, \quad y = r \sin\varphi;$$

and substituting these values in the equation of the surface, we shall have for the equation of the intersection referred to the co-ordinates $\varphi$, $z$, and $r$,

$$Mz^2 + r^2 (M' \sin^2\varphi + M'' \cos^2\varphi) + L = o.$$

This equation will represent different ellipses according to the value of $\varphi$. If $M' = M''$, the axes AC and AD become equal, the angle $\varphi$ disappears, and we have simply

$$Mz^2 + M'r^2 + L = o.$$

Every plane passing through the axis of $z$, will intersect the surface in curves which will be equal to each other, and to the principal sections in the planes of $xz$ and $yz$. The third principal section becomes the circumference of a circle, and all the sections made by parallel planes will also be circles, but with unequal radii. The surface may therefore be generated by the revolution of the ellipse BC or BD around the axis of $z$.

This surface is called an *Ellipsoid of Revolution.*

359. The supposition of $M = M'$, or $M = M''$, would have given an ellipsoid of revolution around the axes of $x$ and $y$.

360. If $M = M' = M''$ the three axes A B C are equal, and the equation of the surfaces becomes

$$z^2 + y^2 + x^2 + \frac{L}{M} = o,$$

which is the equation of a *Sphere*.

361. Generally, as the quantities M, M′, M″, diminish, L remaining constant, the axes which correspond to them augment, and the ellipsoid is elongated in the direction of the axis which increases. If one of them, as M″, becomes zero, the corresponding axis becomes infinite, and the ellipsoid is changed into a *cylinder*, whose axis is the axis of $z$, and whose equation is

$$Mz^2 + M'y^2 + L = o.$$

The base of this cylinder is the ellipse BD. (See figure, Art. 356.)

362. If $M'' = o$, and $M = M'$, the ellipse BD becomes a circle, and the cylinder becomes a *right cylinder with a circular base.* This is the cylinder known in Geometry.

363. Finally, if $M'' = o$, and $M' = o$, the equation reduces to

$$Mz^2 + L = o,$$

which gives

$$z = \pm\sqrt{\frac{-L}{M}}.$$

This equation represents two planes, parallel to that of $xy$, and at equal distances above and below it.

CASE II.—M *positive*, M′ *and* M″ *negative.*

364. In this case the equation of the surface becomes

$$Mz^2 - M'y^2 - M''x^2 + L = o,$$

and the equations of the intersections parallel to the co-ordinate planes are

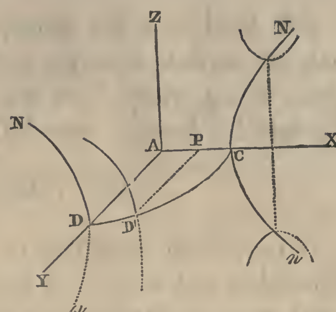
$$Mz^2 - M'y^2 - M''\alpha^2 + L = o,$$
$$Mz^2 - M''x^2 - M'\beta^2 + L = o,$$
$$M'y^2 + M''x^2 - M\gamma \quad - L = o.$$

The two first represent *hyperbolas;* the last is an *ellipse.* The sections parallel to the planes of $xz$ and $yz$ are always real. The section parallel to $xy$ will be always real when L is positive. If L be negative and equal to $-$ L′, it will be imaginary for all values of $\gamma$, which make the quantity ($L' - M\gamma$) positive: when we have $L' - M\gamma = o$, it reduces to a point. Thus, in these two cases, the surface extends indefinitely in every direction, but its form is not the same.

365. Making $\alpha = o$, $\beta = o$, $\gamma = o$, we have for the equations of the principal sections,

$$Mz^2 - M'y^2 + L = o,$$
$$Mz^2 - M''x^2 + L = o,$$
$$M'y^2 + M''x^2 - L = o.$$

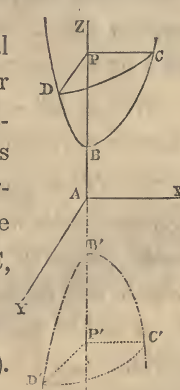
When L is positive, the two first, which are hyperbolas, have the axis of $z$ for a conjugate axis, and are situated as in the figure. Every plane parallel to the plane of $xy$ produces sections which are ellipses.




366. Making the co-ordinates successively equal to zero, we may find the expressions for the semi-axes, as in Art. 357; and representing them respectively by A, B, C $\sqrt{-1}$, and introducing them in the equation of the surface, it becomes

$$A^2B^2z^2 - A^2C^2y^2 - B^2C^2x^2 + A^2B^2C^2 = o. \quad (1).$$

367. When L is negative, the principal sections, which are hyperbolas, have BB′ for the transverse axis; the surface is imaginary from B to B′, and the secant planes between these limits do not meet the surface. In this case, the semi-axes will be found to be A $\sqrt{-1}$, B $\sqrt{-1}$, and C, and the equation of the surface becomes




$$A^2B^2z^2 - A^2C^2y^2 - B^2C^2x^2 - A^2B^2C^2 = o. \quad (2).$$

The surfaces represented by equations (1) and (2) are called *Hyperboloids*. In the first, two of the axes are real, the third being imaginary; and in the second, two are imaginary, the third being real.

368. If M′ = M″, we have A = B, these two surfaces become *Hyperboloids of Revolution* about the axis of $z$.

369. If $M'' = o$, the corresponding axis becomes infinite and the surface becomes a cylinder perpendicular to the plane of $zy$, whose base is a hyperbola. The situation of the cylinder depends upon the sign of L. Its equation is

$$Mz^2 - M'y^2 + L = o.$$

If L diminish, positively or negatively, the interval BB′ diminishes, and when $L = o$, we have $BB' = o$. The principal sections in the planes of $zx$ and $yz$ become straight lines, and the surfaces reduce to a right cone with an elliptical base, having its vertex at the origin of co-ordinates. In this case, we have the equation

$$Mz^2 - M'y^2 - M''x^2 = o.$$

Sections made by planes parallel to the planes of $xz$ and $yz$, are still hyperbolas, which have their centre on the axis of $y$ or $x$.

370. If $M'' = o$, the cone reduces to two planes perpendicular to the planes of $yz$, and passing through the origin.

371. The cone which we have just considered, is to the hyperboloids what asymptotes are to hyperbolas, and the same property may be demonstrated to belong to them, which has been discovered in Art. 259. If we represent by $z$ and $z'$ the respective co-ordinates of the cone and hyperboloid, we shall have

$$z^2 = \frac{M'y^2 + M''x^2}{M}, \quad z'^2 = \frac{M'y^2 + M''x^2 - L}{M},$$

which gives

$$z - z' = \frac{L}{M(z + z')}.$$

The sign of this difference will depend upon that of L; hence, the cone will be interior to the hyperboloid, when L is positive, and exterior to it, when L is negative. The difference $z - z'$ will constantly diminish, as $z$ and $z'$ increase, hence the cone will continually approach the hyperboloid, without ever coinciding with it.

### *Of Surfaces of the Second Order which have no Centre.*

372. Let us resume the equation

$$Mz^2 + M'y^2 + M''x^2 + Hz + H'y + H''x = o. \qquad (2).$$

We have seen (Art. ), that this equation represents surfaces which have no centre when M, M', or M'' is zero. As these three quantities cannot be zero at the same, since the equation would then reduce to that of a plane (Art. 112), we may distinguish two cases;

1st case, M'' equal to zero.

2nd case, M'' and M' equal to zero.

#### CASE I.—M'' *equal to zero.*

373. The above equation under this supposition reduces to

$$Mz^2 + M'y^2 + Hz + H'y + H''x = o.$$

If we refer this equation to a new system of co-ordinates taken parallel to the old, we may give such values to the independent constants as to cause the co-efficients H' and H'' to disappear, (Art. 350). The equation will then become

$$Mz^2 + M'y^2 + H''z = o.$$

374. The sections parallel to the co-ordinate planes are,

$$Mz^2 + M'y^2 + H''\alpha = o,$$
$$Mz^2 + H''x + M'\beta^2 = o,$$
$$M'y^2 + H''x + M\gamma^2 = o.$$

The two first represent *parabolas*, and are always real. The third equation will represent an ellipse or hyperbola, according to the sign of M and M'.

375. The principal sections are

$$Mz^2 + M'y^2 = o, \quad Mz^2 + H''x = o, \quad M'y^2 + H''x = o.$$

The first of these equations will represent a point, or two straight lines, according to the sign of M'. The two others represent parabolas.

376. Let us suppose M and M' positive, the sections parallel to the plane of $yz$, and whose equation is

$$Mz^2 + M'y^2 + H''\alpha = o,$$

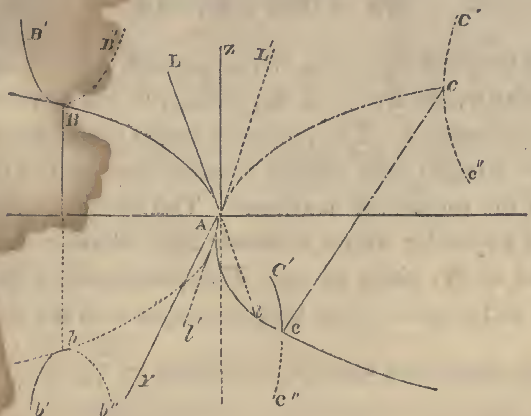
will only be real when H'' and $\alpha$ have contrary signs. The surface, therefore, will extend indefinitely on the positive side of the plane of $yz$, when H' is negative, and on the negative side when H' is positive.

377. If M' be negative, the equations of the principal sections are

$$Mz^2 - M'y^2 = o, \quad Mz^2 + H''x = o, \quad M'y^2 - H''x = o.$$

The two last represent parabolas, having their branches extending in opposite directions, and their vertex at the

origin A. The sections parallel to the plane of $yz$, will be the hyperbolas B, B′, B″, C, C′, C″.




The surfaces which we have just discussed are called *Paraboloids*.

CASE II.—M′ *and* M″ *equal to zero.*

378. Equation (2) under this supposition reduces to

$$Mz^2 + Hz + H'y + H''x = o.$$

Moving the origin of co-ordinates so as to cause the term H$z$ to disappear, this equation becomes

$$Mz^2 + H'y + H''x = o.$$

The principal sections of this surface are

$$Mz^2 + H'y = o, \quad Mz^2 + H''x = o, \quad H'y + H''x = o,$$

and the sections parallel to the co-ordinate planes

$$Mz^2 + H'y + H''\alpha = o,$$
$$Mz^2 + H''x + H'\beta = o,$$
$$H'y + H''x + M\gamma^2 = o.$$

The two first equations of the parallel sections represent parabolas which are equal and parallel to the corresponding principal sections. The sections parallel to the plane of $xy$, are two straight lines parallel to each other and to intersection of the surface by this plane. The surface is, therefore, that of a cylinder with a *parabolic* base, whose elements are parallel to the plane of $xy$. The projections of these elements on the plane of $xy$, make an angle with the axis of $x$, the trigonometrical tangent of which is $-\frac{H''}{H}$.

*Of Tangent Planes to Surfaces of the Second Order.*

379. A tangent plane to a curved surface at any point is the *locus* of all *lines* drawn tangent to the surface at this point.

380. Let us seek the equation of a tangent plane to surfaces of the second order. Resuming the equation

$$Az^2+A'y^2+A''x^2+Byz+B'xz+B''xy+Cz+C'y+C''x+F=o,$$

and transforming it, so as to cause the terms containing the rectangle of the variables to disappear, we have

$$Az^2 + A'y^2 + A''x^2 + Cz + C'y + C''x + F = o. \quad (1.)$$

Let $x''$, $y''$, $z''$, be the co-ordinates of the point of tangency, they must satisfy the equation of the surface, and we have

$$Az''^2 + A'y''^2 + A''x''^2 + Cz'' + C'y'' + C''x'' + F = o.$$

The equations of any straight line drawn through this point are (Art. ),

$$x - x'' = a\,(z - z''), \quad y - y'' = b\,(z - z'').$$

For the points in which this line meets the surface, these equations subsist at the same time with that of the surface. Combining them, we have

$$A(z + z'')\,(z - z'') + A'\,(y + y'')\,(y - y'') + A''\,(x + x'')\,(x - x'') + C\,(z - z'') + C'\,(y - y'') + C''\,(x - x'') = o.$$

Putting for $y - y''$ and $x - x''$, their values drawn from the equations of the straight line, we have

$$\{A(z + z'') + A'b\,(y + y'') + A''a\,(x + x'') + C + C'b + C''a\}\,(z - z'') = o.$$

This equation is satisfied when $z - z'' = o$, which gives $z = z''$, $x = x''$, and $y = y''$. Suppressing $(z - z'')$, we have

$$A(z + z'') + A'b\,(y + y'') + A''a\,(x + x'') + C + C'b + C''a = o.$$

This equation determines the co-ordinates of the second point in which the line meets the surface. But if this line becomes a tangent, the co-ordinates of the second point will be the same as those of the point of tangency, we shall have therefore

$$x = x'',\ y = y'',\ z = z'',$$

which gives

$$2Az'' + 2A'by'' + 2A''ax'' + C + C'b + C''a = o,$$

for the *condition that a straight line be tangent to a surface of the second order.* Since this equation does not determine the two quantities $a$ and $b$, it follows that an infinite number of lines may be drawn tangent to this surface at any point. If $a$ and $b$ be eliminated by means of their values taken from the equations of the straight line, the resulting equation will be that of the *locus* of these tangents. The elimination gives

$$(2Az'' + C)(z - z'') + (2A'y'' + C')(y - y'') \\ + (2A''x'' + C'')(x - x'') = o;$$

and since this equation is of the first degree with respect to $x$, $y$ and $z$, the *locus* of these tangents is a *plane* which is itself tangent to the surface.

**381.** Developing this last equation, and making use of equation (1), the equation of the tangent plane may be put under the form

$$(2Az'' + C)z + (2A'y'' + C')y + (2A''x'' + C'')x \\ + Cz'' + C'y'' + C''x'' + 2F = o.$$

**382.** For surfaces which have a centre, C, C′, C″, are zero, and the equation of their tangent plane becomes

$$Azz'' + A'yy'' + A''xx'' + F = o.$$

# APPENDIX.

## TRIGONOMETRICAL FORMULÆ.

1. $\text{tang } a = \text{R} \frac{\sin a}{\cos a}$.

2. $\text{cotang } a = \text{R} \frac{\cos a}{\sin a}$.

3. $\text{sec. } a = \frac{\text{R}^2}{\cos a}$.

4. $\text{co-sec } a = \frac{\text{R}^2}{\sin a}$.

5. $\sin (a + b) = \sin a \cos b + \sin b \cos a$.

6. $\cos (a + b) = \cos a \cos b - \sin a \sin b$.

7. $\sin (a - b) = \sin a \cos b - \sin b \cos a$.

8. $\cos (a - b) = \cos a \cos b + \sin a \sin b$.

9. $\text{tang } (a + b) = \frac{\text{tang } a + \text{tang } b}{1 - \text{tang } a \text{ tang } b}$.

10. $\text{tang } (a - b) = \frac{\text{tang } a - \text{tang } b}{1 + \text{tang } a \text{ tang } b}$.

11. $\text{tang } 2a = \frac{2 \text{ tang } a}{1 - \text{tang}^2 a}$.

12. $\frac{\sin a + \sin b}{\sin a - \sin b} = \frac{\text{tang } \frac{1}{2} (a + b)}{\text{tang } \frac{1}{2} (a - b)}$.

13. $\sin^2 \frac{1}{2} a = \frac{1 - \cos a}{2}$.

14. $\cos^2 \frac{1}{2} a = \frac{1 + \cos a}{2}$.

15. $\operatorname{tang}^2 \frac{1}{2}a = \dfrac{1 - \cos a}{1 + \cos a}.$

16. $\sin 2a = 2 \sin a \cos a.$

17. $\sin^2 a = \dfrac{\operatorname{tang}^2 a}{1 + \operatorname{tang}^2 a}.$

18. $\cos^2 a = \dfrac{1}{1 + \operatorname{tang}^2 a}.$

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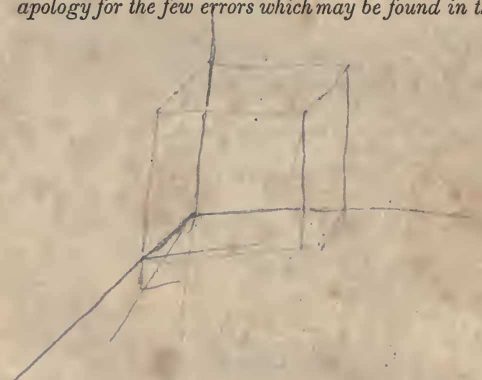
NOTE.—*The distance of the Translator from the Press, must be his apology for the few errors which may be found in the work.*



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